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Presentation Functions, Fixed Points, and a Theory of Scaling Function Dynamics

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Presentation functions provide the time-ordered points of the forward dynamics of a system as successive inverse images. They generally determine objects constructed on trees, regular or otherwise, and immediately determine a functional form of the transfer matrix of these systems. Presentation functions for regular binary trees determine the associated forward dynamics to be that of a period doubling fixed point. They are generally parametrized by the trajectory scaling function of the dynamics in a natural way. The requirement that the forward dynamics be smooth with a critical point determines a complete set of equations whose solution is the scaling function. These equations are compatible with a dynamics in the space of scalings which is conjectured, with numerical and intuitive support, to possess its solution as a unique, globally attracting fixed point. It is argued that such dynamics is to be sought as a program for the solution of chaotic dynamics. In the course of the exposition new information pertaining to universal mode locking is presented.

KEY WORDS: Scaling; thermodynamics; period doubling; mode locking; dynamical systems; chaos; renormalization group.

1. INTRODUCTION

The attempts to understand the full microscopic structure of chaotic dynamical motion succeeded through renormalization group-like treatments for a variety of transitional phenomena. The upshot of that work is a complicated delineation of the parameter space as well as the phase space marked by scaling phenomena. An important idea in that work was that the *dynamics* under the fixed-point map inherits from the fixed-point equation determining it a rich set of scaling symmetries. Indeed, the trajectory *scaling function* determined from the fixed point allows the full deter-

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mination of the orbit (and its Cantor set closure), or, Fourier transformed, the power spectrum given some low-resolution scale-fixing data. This information is embedding-free (an invariant—indeed the maximal invariant) and allows such computations whatever the embedding dimension of the phenomenon. The objects determined by these scaling functions are true "multifractals" in the present parlance, but of the richest variety with an infinity of scales.

As one pursues more chaotic, higher-dimensional strange attractors, it appears unlikely that a fully microscopic scaling function theory can be offered. At least provisionally, one has turned attention to various dimension-related notions of mere thermodynamic descrition. This raises questions as to just what these degenerate objects and their interrelations depend upon, how they are to be calculated, and their extensions and generalizations. A particular question of outstanding interest is the deduction of the numerically well-established dimension of the "gaps" in the set of all mode-locked intervals of quasiperiodic motion—the computation of average quantities from a highly complex microscopic distribution.

The crux of the above question is that, given a highly variegated set of microscopic scalings which determine average quantities as a subset of this full information, can one undo the labor of their extraction and find a better-behaved, simpler substrate from which the averages easily follow? The goal of this paper is to first present such machinery—that of so-called "presentation functions"—and then learn how fixed-point dynamics, scallings, and these new objects are all interrelated.

The plan of the paper is as follows. In the second section I extract the presentation function of period doubling dynamics from the fixed-point equation of the latter, and discern in it the schema of organizing arbitrary objects constructed on binary trees. In Section 3 I show how to immediately write down the thermodynamics given a presentation function by explicitly writing down a functional operator form of the "transfer matrix." I illuminate these notions by way of examples in Section 4 and present numerical evidence as to how mode locking thermodynamics might be derived. In Section 5 the notion of generalizations of the machinery to arbitrary trees, complete or "pruned," is developed. In Section 6 I invert the exposition and discover how the period doubling dynamics for a given tree can be obtained from its prior specified presentation function, thereby further exposing the "heart" of the mode locking problem and paving the way toward a new method of obtaining the fixed-point function. In Section 7 I relate presentation functions to scaling functions, the latter a special parameterization of the former. In Section 8 I explicitly write down all successive approximations to the computation of the scaling function. In Section 9 I present a dynamics in the space of scalings that (as a numerical observation in higher orders of approximation) possesses the solution of the equations of Section 8 as a globally stable fixed point and suggest that a new possibility for the rigorous proof of the isolated solution to the fixed-point equation is at hand. In a final section, I take stock of this offering and contemplate its meaning and possible future directions.

2. PRESENTATION FUNCTIONS

The dynamics of the period doubling fixed point[1] provides a scheme for the organization of objects definned by a regular binary tree. The fixed point obeys the equation

$$\alpha g^2 \alpha^{-1} = g; \qquad g(0) = 1 \tag{2.1}$$

where α denotes the linear transformation of multiplication by α. The function g is understood to possess (usually a quadratic) critical point at $x=0$.

Denoting the critical value by x_0,

$$x_0 = g(0) = 1 \tag{2.2}$$

define the nth image under g of x_0 by x_n:

$$x_n = g^n(x_0) = g^{n+1}(0) \tag{2.3}$$

Observe that

$$x_{2n+1} = g^{2n+2}(0) = \alpha^{-1} g^{n+1} \alpha(0) = \alpha^{-1} x_n \tag{2.4}$$

and

$$x_{2n} = g^{2n+1}(0) = g^{-1}(x_{2n+1}) = g^{-1}\alpha^{-1} x_n = (\alpha g)^{-1}(x_n) \tag{2.5}$$

Since g is unimodal, there are *two* inverses of g. However, g alternately maps a central domain to a right domain, and g^2 maps within either of these domains. Thus, x_{2n} is always in the right domain that includes the critical value x_0. Thus, g^{-1} of (5) is the right inverse of g, that is, the inverse of the right half of g restricted to a domain strictly excluding the critical point, and so of bounded nonlinearity.

Let us denote the two results, (2.4) and (2.5), by

$$x_{2n+\varepsilon} = F_\varepsilon(x_n), \qquad \varepsilon = 0, 1 \tag{2.6}$$

so that

$$F_0(x) = (\alpha g)^{-1}(x) \tag{2.7}$$

$$F_1(x) = \alpha^{-1} x \tag{2.8}$$

Further, let us regard F_0 and F_1 as the two inverses of a "unimodal" map E

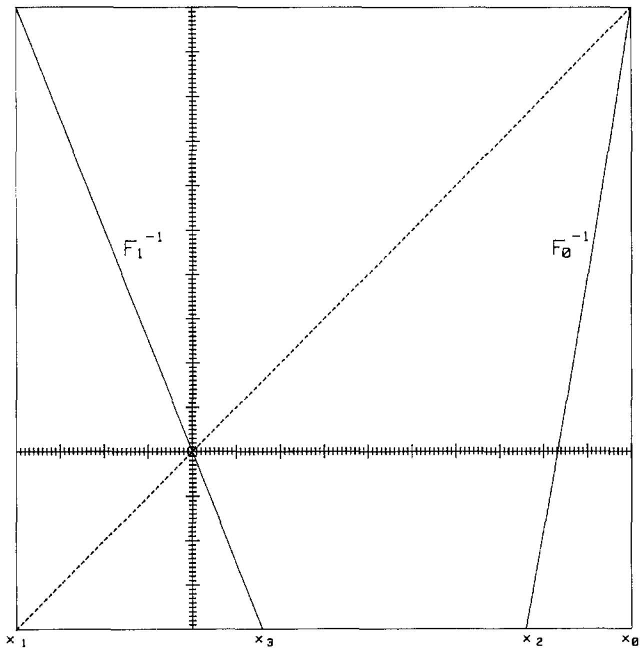

Fig. 1. The graph of the expanding map E for the quadratic period doubling fixed point. $F_1^{-1}=\alpha x$, $F_0^{-1}=\alpha g(x)$. The Cantor set is covered by the range $[x_1, x_3]$ of F_1 and $[x_2, x_0]$ of F_0.

(it can be verified that F_0 contracts, so that E is expanding). E is deicted in Fig. 1, and is defined on the two disjoint intervals $[x_1, x_3]$ and $[x_2, x_0]$. It is immediately verified from (2.1) and the content of (2.4) and (2.5) that

$$Eg^2 = gE \tag{2.9}$$

that is, E conjugates g^2 to g. Thus, the period doubling attractor, the closure of the orbit of x_0, is that Cantor set for which the center piece is mapped to the entire set by α and the right-hand piece is mapped to the entire set by αg.

Notice that E is defined on two intervals; E^2 is defined on four disjoint intervals, etc. The domain of E^n is a cover of the Cantor set by 2^n disjoint intervals. The intersection of the domains of E^n over all n is the Cantor set itself. E thus "presents" the Cantor set and is called its "presentation function."[2] Had we replaced αg by an approximating *linear* map, we see by the above that a two-scale Cantor set would have been constructed.

[2] The ideas of presentation functions are sprinkled in the literature. The name arose in discussions between D. Sullivan and myself. To my knowledge, its role here in carrying full dynamical information is new.

Let us consider E in the spirit of a dynamical system itself. Specifically, let us consider the set of all inverse images of x_0 under E. Denote the index r of an $x_r^{(n)}$ by its binary expansion

$$r = \varepsilon_n \cdots \varepsilon_1, \qquad \varepsilon_i = 0, 1 \tag{2.10}$$

so that $r = 0,..., 2^n - 1$ for any point in the set of nth inverse images. By (2.10) we immediately see that

$$x^{(n)}_{\varepsilon_n \cdots \varepsilon_1} = F_{\varepsilon_1} \circ F_{\varepsilon_2} \circ \cdots \circ F_{\varepsilon_n}(x_0) \tag{2.11}$$

Since

$$x_0 = x_{2x0} = F_0(x_0) \tag{2.12}$$

x_0 is a fixed point of F_0, so that, by (2.11), the first 2^{n-1} inverses of the nth set make up precisely the $(n-1)$th set of inverses. Thus,

$$x^{(n+r)}_{\underset{r}{0 \cdots 0}\varepsilon_n \cdots \varepsilon_1} = x^{(n)}_{\varepsilon_n \cdots \varepsilon_1} \tag{2.13}$$

and the superscript (n) is superfluous: a point with the same evaluated index (leading 0 ε's) is the same at all levels possessing it.

But, x_n as defined in (3) is the nth image under g of x_0. *Thus, the inverse under E of a given index is precisely that iterate of the forward dynamics of g.* I will develop this observation later: namely, that to a given presentation function E, one can associate a period doubling fixed point, the dynamics of which agrees with the inverses under E. That is, to an *a priori* specified Cantor set, one can associate a dynamics that possesses the Cantor set as its attractor. Before doing so, however, let us investigate the thermodynamics of the Cantor set given its presentation function.

3. THE THERMODYNAMICS OF PRESENTATION FUNCTIONS

By thermodynamics I mean the deduction of variables and their relations to one another that in some well-defined sense are averages of exponential quantities defined microscopically on a set.[3] The exponential quantities here are the lengths of intervals covering a set.

Notice by (2.11) that an $x^{(n)}$ is obtained by n contractive mappings, so that $x^{(n)}\varepsilon_n \cdots \varepsilon_1$ is weakly dependent upon the higher indexed (leftmost) ε's.

[3] These ideas appear in the classical literature, and can be found in refs. 2 and 3. The notation here employed and the important focus on return times appears in ref. 4. The above work expositing Markov graphs for golden mean rotation should be understood as a serious elaboration upon ref. 5.

We are thus naturally led to approximations (not always actual covers in generalizations to follow) to the set by a set of 2^n intervals with each interval defined by the endpoints

$$x^{(n+1)}_{0\,\varepsilon_n\cdots\varepsilon_1} \quad \text{and} \quad x^{(n+1)}_{1\,\varepsilon_n\cdots\varepsilon_1} \tag{3.1}$$

Let us then denote an indexed nth-level interval length by

$$\Delta^{(n)}(\varepsilon_n\cdots\varepsilon_1) = x^{(n+1)}_{0\,\varepsilon_n\cdots\varepsilon_1} - x^{(n+1)}_{1\,\varepsilon_n\cdots\varepsilon_1} \tag{3.2}$$

By (2.11), we can estimate

$$\begin{aligned}\Delta^{(n)}(\varepsilon_n\cdots\varepsilon_1) &= F_{\varepsilon_1}\cdots F_{\varepsilon_n}F_0(x^{(0)}_0) - F_{\varepsilon_1}\cdots F_{\varepsilon_n}F_1(x^{(0)}_0)\\ &\approx F'_{\varepsilon_1}(F_{\varepsilon_2}\cdots)\,F'_{\varepsilon_2}(F_{\varepsilon_3}\cdots)\cdots F'_{\varepsilon_r}(F_{\varepsilon_{r+1}}\cdots)\cdots\end{aligned} \tag{3.3}$$

The idea of (3.3) is that for n large, asymptotically each of the derivatives is taken at an argument insensitive to the highest indexed ε's so that each F_ε is asymptotically linear over the required range. This follows for F_ε that are differentiable, contractive, and of bounded nonlinearity. [A weakening of contractive is allowed to include $|F'(x)| = 1$ at the boundary of definition.] The $\Delta^{(n)}$ will then be bounded by exponentials in n, that is, loosely,

$$|\Delta^{(n)}(\varepsilon_n\cdots\varepsilon_1)| \sim e^{-nh(\varepsilon_n\cdots\varepsilon_1)} = e^{-H(\varepsilon_n\cdots\varepsilon_1)} \tag{3.4}$$

where, by the above reasoning, it can easily be seen that h has increasingly weak dependence upon *low-index* ε's. (I shall return to this matter—generally the matter of scaling—later.)

Accordingly, in strict analogy to statistical mechanics, it is natural to consider the sum

$$\sum_{\{\varepsilon\}} |\Delta^{(n)}(\varepsilon_n\cdots\varepsilon_1)|^\beta \sim \sum_{\{\varepsilon\}} e^{-\beta H(\varepsilon_n\cdots\varepsilon_1)} \tag{3.5}$$

Since H is extensive in n, it is seen that the sum in (3.5) is the canonical partition sum for a statistical mechanical system of n "spins" ε_i one each at each lattice site i of a one-dimensional array. With "interactions" falling off with site separation, one expects the thermodynamic limit

$$\sum_{\{\varepsilon\}} |\Delta^{(n)}(\varepsilon_n\cdots\varepsilon_1)|^\beta \underset{n\to\infty}{\sim} 2^{-nF(\beta)} \tag{3.6}$$

to go through (in the sense of logarithms). This "free energy" $F(\beta)$ expresses the thermodynamic relation between the variables F and β.

Notice that β_H such that

$$F(\beta_H) = 0 \tag{3.7}$$

is "special" in that the sum has no exponential growth as the covering intervals are successively refined. Thus, β_H is to be identified with the Hausdorff dimension of the set. (If the "approximations" are as optimal as their definition suggests them to be, β_H *will* be the Hausdorff dimension.) Thus, β is related to "generalized" dimensions of the set.

Let us now proceed formally. Defining

$$\lambda(\beta) = 2^{-F(\beta)} \tag{3.8}$$

(3.6) asserts that

$$\sum |\Delta^{(n)}|^\beta \sim \lambda^n \tag{3.9}$$

and we seek an eigenvalue equation determining $\lambda(\beta)$. This is elementary to obtain from the F_ε; all we need do is use (3.3) in (3.9), writing the latter as an iterated sum:

$$\sum_{\{\varepsilon\}} |\Delta^{(n)}(\varepsilon_n \cdots \varepsilon_1)|^\beta \sim \cdots \sum_{\varepsilon_r} |F'_{\varepsilon_r}(F_{\varepsilon_{r+1}} \cdots)|^\beta \sum_{\varepsilon_{r-1}} |F'_{\varepsilon_{r-1}}(F_{\varepsilon_r} F_{\varepsilon_{r+1}} \cdots)|^\beta \cdots \tag{3.10}$$

Notice that the sum over ε_{r-1} and all lower-ε sums to its right depend only upon $F_{\varepsilon_r} F_{\varepsilon_{r+1}} \cdots$ defined by the outer sums. Accodingly, writing

$$\psi_{r-1}(F_{\varepsilon_r} F_{\varepsilon_{r+1}} \cdots) = \sum_{\varepsilon_{r-1}} |F'_{r-1}(F_{\varepsilon_r} F_{\varepsilon_{r+1}} \cdots)|^\beta \sum_{\varepsilon_{r-2}} |F'_{\varepsilon_{r-2}}(F_{\varepsilon_{r-1}}(F_{\varepsilon_r} \cdots))|^\beta \cdots \tag{3.11}$$

we then have

$$\psi_r(F_{\varepsilon_{r+1}} \cdots) = \sum_{\varepsilon_r} |F'_{\varepsilon_r}(F_{\varepsilon_{r+1}} \cdots)|^\beta \, \psi_{r-1}(F_{\varepsilon_r}(F_{\varepsilon_{r+1}} \cdots)) \tag{3.12}$$

Denoting any possible point $F_{\varepsilon_{r+1}} F_{\varepsilon_{r+2}} \cdots$ by x, we have that (3.12) reads

$$\psi_r(x) = \sum_\varepsilon |F'_\varepsilon(x)|^\beta \psi_{r-1}(F_\varepsilon(x)) \tag{3.13}$$

Since (3.13) is a linear transformation, $\psi_r(x)$ asymptotically in r behaves as

$$\psi_r(x) \sim \lambda^r \psi(x) \tag{3.14}$$

where λ is the largest eigenvalue obeying

$$\lambda \psi(x) = \sum_\varepsilon |F'_\varepsilon(x)|^\beta \, \psi(F_\varepsilon(x)) \tag{3.15}$$

Since the sum of (3.10) is $\psi_n \sim \lambda^n$, the λ of (3.15) is λ of (3.9) and (3.8), so that (3.15) is the desired eigenvalue equation for $\lambda(\beta)$.

[Although $\psi(x)$ by its definition was defined only for x in the "attractor" which can be proper (Cantor) subset of the interval, with F_ε differentiable enough, (3.15) determines its extension to all x.]

Thus, given the fixed point $g(x)$ for quadratic period doubling, (2.7) and (2.8) entered in (3.15) explicitly determines the equation for the thermodynamics of the period doubling attractor. It is useful, however, to write down and solve (3.15) for simpler presentation functions (i.e., when the F_ε are explicitly available). Accordingly, let us analyze both a trivial problem (a two-scale Cantor set) and highly nontrivial one (the parameter axis for subcritical quasiperiodic motion).[4]

4. EXAMPLES OF THERMODYNAMICS

1. Write $\alpha^{-1} = -\sigma_1$, so that

$$F_1(x) = -\sigma_1 x \tag{4.1}$$

and replace F_0 of Fig. 1 by

$$F_0(x) = 1 + \sigma_0(x-1) \tag{4.2}$$

with

$$0 < \sigma_0 < \sigma_1 < 1$$

Then (3.15) becomes

$$\lambda\phi(x) = \sigma_1^\beta \psi(-\sigma_1 x) + \sigma_0^\beta \psi(1 + \sigma_0(x-1)) \tag{4.3}$$

(4.3) clearly possesses the solution

$$\psi = \text{const}, \qquad \lambda = \sigma_1^\beta + \sigma_0^\beta \tag{4.4}$$

It is also easy to see that for each $n > 0$ there is a ψ_n, a polynomial of degree n in x, and $\lambda_n = \sigma_1^\beta(-\sigma_1)^n + \sigma_0^{\beta+n}$. However, the eigenvalue (4.4), λ_0, exceeds all these λ_n, and so (4.4) is the solution.

2. Consider the binary Farey tree shown in Fig. 2, constructed by placing

$$\frac{p}{q} \oplus \frac{p'}{q'} = \frac{p+p'}{q+q'}$$

between every pair of previously determined (*all* prior levels) fractions. It is easy to show that the nth layer of the tree consists precisely of all those fractions

$$[c_1, \ldots, c_k] = \cfrac{1}{c_1 + \cfrac{1}{c_2 + \cdots}}, \qquad c_k \geqslant 2 \tag{4.5}$$

[4] An initial exposition of these ideas can be found in ref. 6. More examples of this machinery and a discussion of the spectrum of (3.15) appear in ref. 7.

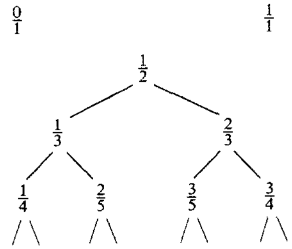

Fig. 2. Binary Farey tree.

for which

$$\sum_{1}^{k} c_i = n+2 \tag{4.6}$$

Define now a pair of transformations that determine the $(n+1)$th layer from the nth:

$$\begin{aligned} F_0:\quad & [c_1,..., c_k] \to [c_1+1, c_2,..., c_k] \\ F_1:\quad & [c_1,..., c_k] \to [1, c_1,..., c_k] \end{aligned} \tag{4.7}$$

It follows from (4.6) that (4.7) will inded accomplish its defined task. By (4.7),

$$F_1([c_1,..., c_k]) = \cfrac{1}{1+\cfrac{1}{c_1+\cdots}} = \frac{1}{1+[c_1,..., c_k]}$$

i.e.,

$$F_1(x) = \frac{1}{1+x} \tag{4.8}$$

and similarly,

$$F_0(x) = \frac{x}{1+x} \tag{4.9}$$

We now regard these F_ε as the inverses of E drawn in Fig. 3, with F_0 the left inverse and F_1 the right inverse.

Starting at $x_0^{(0)} = 1/2$, $x_0^{(1)} = F_0(1/2) = 1/3$, $x_1^{(1)} = F_1(1/2) = 2/3$, and, generally, the 2^n nth inverses of 1/2 are precisely the nth layer of the Farey tree. Notice that $F_0'(0) = 1$, so that each F_ε becomes marginally noncontractive at one endpoint of its domain. The intervals computed according to (3.2) are precisely the spacings between the two descendants at level n of a

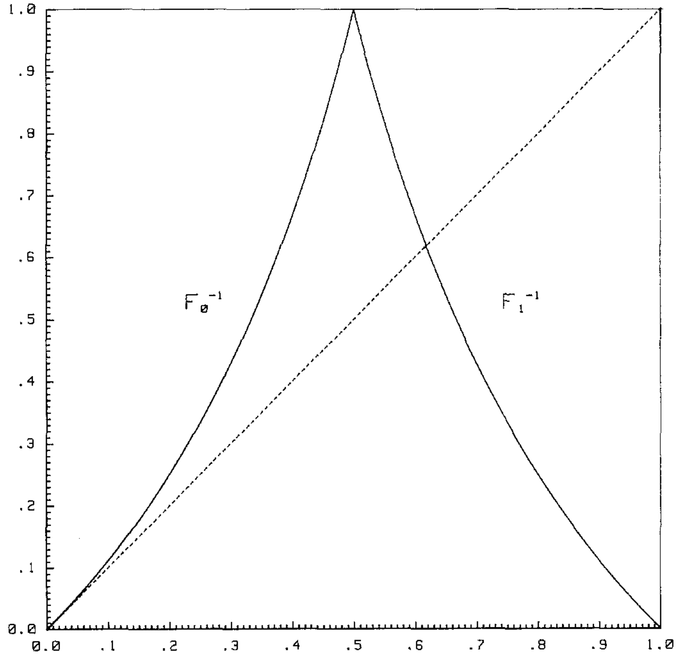

Fig. 3. The presentation function for the Farey model, Eqs. (4.8), (4.9).

level-$(n-1)$ parent (for example, $2/5 - 1/4$, the children of $1/3$ in Fig. 2). Since the tree determines all rationals, the "attractor" that E of Fig. 3 determines is the entire interval $[0, 1]$ partitioned at each level in a highly nontrivial fashion. The set of nth-level intervals now is a representative "subcovering" of the entire interval. As pointed out elsewhere, the study of these intervals is intimately related to the full devil's straircase of mode locking intervals for two coupled nonlinear ocillators, so that the results are of high physical interest.[5]

As mentioned at the end of Section 2, it is possible to construct a period doubling fixed point the *dynamics* of which is identical to the inverses of this E. I have elsewhere determined this so-called "Farey map," which has the property that it possesses a periodic orbit of length 2^n, one

[5] I first invented this "Farey model," itself a correct treatment of subcriticality, as a trial problem to understand the organization of the centroids of critical mode locking intervals. I learned of the critical problem and its suggestive scaling function from Cvitanovic. The essence of the idea that that organization is accomplished by a period doubling fixed point (and hence the organization of any complete binary tree) was developed in 1984 in my partially circulated and unpublished paper, "The renormalization of the Farey map." The present paper now renders that paper essentially defunct.

for each n, and of identical marginal stability -1, which are precisely the layers of the Farey tree. I shall explore this general matter in the next section.

Utilizing (4.8) and (3.15), we arrive at the equation

$$\lambda\psi(x) = (1+x)^{-2\beta}\left[\psi\left(\frac{x}{1+x}\right) + \psi\left(\frac{1}{1+x}\right)\right] \tag{4.10}$$

Despite the marginally contractive behavior of the F_ε, we know from two other independent (one harder, the other much harder) derivations that (4.10) is correct. There are some available solutions to (4.10).

(i) $\beta = 1 \to \lambda = 1$, $\psi = 1/x$. As anticipated, the set has Hausdorff dimension $\beta_H = 1$ and ψ is nontrivial.

(ii) $\beta = -n/2$, $n = 0, 1, \ldots$; ψ now has polynomial solutions, with the leading ψ a polynominal of degree n. For example, $\lambda(+1/2) = 3$.

(iii) As $\beta \to -\infty$, $\lambda(\beta) \sim \rho^{2\beta}$, $\rho = (\sqrt{5} - 1)/2$, intimately related to the fixed point $\rho = F_1(\rho)$ at which

$$F_1'(\rho) = -\rho^{-2}$$

(iv) For other $\beta < 1$, ψ has branch point at $x = 0$; it is shown in ref. 6 that

$$F \ln(-F) \sim 1 - \beta \qquad \text{as} \quad \beta \to 1 \quad \text{from below}$$

(v) $F(\beta) = 0$, $\beta > 1$.

By property (iv), this problem has an infinite-order phase transition at $\beta = 1$. (More information can be found in Appendix I of ref. 7. In particular, the statistical model is essentially that of a one-dimensional lattice gas with a one-particle saturating logarithmic interaction.)

In order to understand the statistical mechanics connection more fully and to better comprehend the period doubling dynamics equivalent to a given E, we shall have to turn to the notions of scaling functions. I first comment about generalizations.

5. OBJECTS ON COMPLETE AND REGULARLY PRUNED n-ARY TREES

Although we started with period doubling dynamics, it is clear that for any presentation function E we can construct a Cantor set and by (3.15) immediately write down the functional linear operator (the "transfer matrix" of the statistical mechanical analog) whose largest eigenvalue determines the thermodynamic relation of F to β. The simplest generalization away from usual period doubling is the replacement of $F_0 = (\alpha g)^{-1}$ with some other function. More generally, we can write an E with n inverses, each then mapping the entire interval to an appropriate piece of the set. This is simply to replace a binary tree with a general n-ary tree. All of the formulas of the preceding sections remain unchanged, save for each ε, which now ranges from 0 to $n-1$.

However, another important class of objects is not yet covered. Consider, for example E of Fig. 4. The special property is that while any point on the F_1 arc has two inverses, a point on the F_0 arc has only an F_1 inverse. Thus, F_1F_1, F_1F_0 are allowed, but F_0F_0 is not. Thus, only $x_{\varepsilon_n \cdots \varepsilon_1}$ exist for which no two 0 epsilons can be consecutive. This situation is that of the incomplete tree of Fig. 5, which is easily seen to have F_n entries at level n, where $F_{n+1} = F_n + F_{n-1}$ are the Fibonacci numbers, and

$$F_n \sim \rho^{-n}; \qquad \rho^{-1} = \frac{\sqrt{5}+1}{2}$$

Observe that (3.9) implies that $\lambda(0)$ is the geometric growth rate of the number of pieces (or points) at level n, which for E of Fig. 4 should thus be

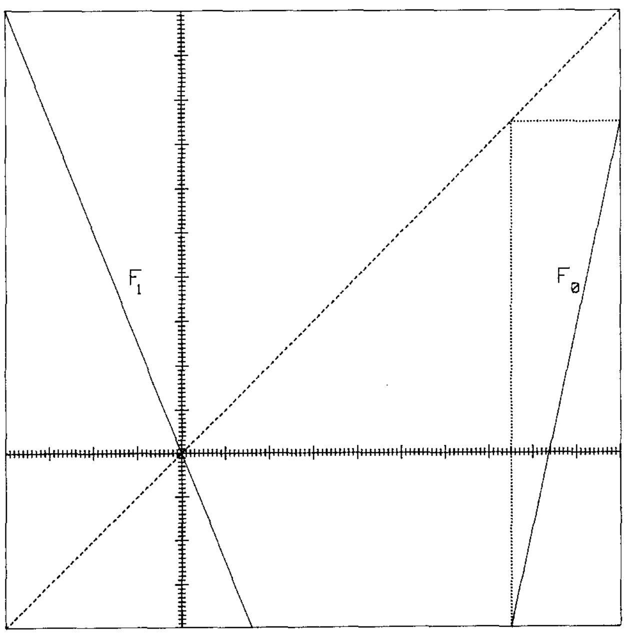

Fig. 4. A presentation function for a golden mean grammar incomplete binary tree: the range of F_0 is disjoint from its domain. (Section 5).

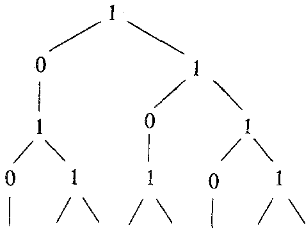

Fig. 5.

ρ^{-1}. For $\beta=0$, (3.15) always possesses $\psi=\text{const}$ as its leading eigenvalue with $\lambda=\sum_\varepsilon 1=2$, and so incompatible with the regularly "pruned" tree of Fig. 5 for E of Fig. 4.

It is easy to find the correct replacement for (3.15). Returning to (3.12), we immediately see that if $F_{\varepsilon_{r+1}}=F_0$, then ε_r cannot take on the value zero. Thus, the generic renaming of

$$F_{\varepsilon_{r+1}}F_{\varepsilon_{r+2}}\cdots=x$$

that led to (3.13) makes it impossible to determine the allowed range of ε summation in (3.13). To solve this problem, we simply backtrack one step and instead by x denote $F_{\varepsilon_{r+2}}F_{\varepsilon_{r+3}}\cdots$, so that (3.15) now reads

$$\lambda\phi(F_{\varepsilon_1}(x))=\sum_{\varepsilon_0}{}' \,|F'_{\varepsilon_0}(F_{\varepsilon_1}(x))|^\beta\,\psi(F_{\varepsilon_0}F_{\varepsilon_1}(x)) \tag{5.1}$$

where $\sum'_{\varepsilon_0}$ means that $F_{\varepsilon_0}F_{\varepsilon_1}$ must be an allowed composition.

Let us now denote the restriction of ψ to the range of F_ε by ψ_ε:

$$\psi_\varepsilon(x)=\psi(F_\varepsilon(x)) \tag{5.2}$$

so that (5.1) now reduces to

$$\lambda\psi_{\varepsilon_1}(x)=\sum_{\varepsilon_0}{}' \,|F'_{\varepsilon_0}(F_{\varepsilon_1}(x))|^\beta\,\psi_{\varepsilon_0}(F_{\varepsilon_1}(x)) \tag{5.3}$$

where again $\sum'_{\varepsilon_0}$ means that only those ε_0 such that $F_{\varepsilon_0}F_{\varepsilon_1}$ is allowed are summed. Equation (5.3) is the correct eigenvalue equation for a "grammar" of strings of ε's of length 2.

Specializing (5.3) to the tree of Fig. 5, we now have

$$\begin{aligned}\lambda\phi_0(x)&=|F'_1(F_0(x))|^\beta\,\psi_1(F_0(x))\\ \lambda\phi_1(x)&=|F'_0(F_1(x))|^\beta\,\psi_0(F_1(x))+|F'_1(F_1(x))|^\beta\,\psi_1(F_1(x))\end{aligned} \tag{5.4}$$

or,

$$\lambda^2\psi_1(x) = |F_0'(F_1(x))|^\beta |F_1'(F_0F_1(x))|^\beta \psi_1(F_0F_1(x)) + \lambda |F_1'(F_1(x))|^\beta \psi_1(F_1(x))$$

which for $\beta = 0$, $\psi_1 = \text{const}$, reads

$$\lambda^2 = 1 + \lambda \to \lambda_> = \frac{\sqrt{5}+1}{2} = \rho^{-1}$$

the correct growth rate.

So far we have treated $2-\varepsilon$ grammars. For restrictions among $n+1$ epsilons, we again turn to (5.12) and extend (5.1) to read

$$\lambda\phi(F_{\varepsilon_1}\cdots F_{\varepsilon_n}(x)) = \sum_{\varepsilon_0}{}' |F_{\varepsilon_0}'(F_{\varepsilon_1}\cdot\cdot F_{\varepsilon_n}(x))|^\beta \psi(F_{\varepsilon_0}\cdots F_{\varepsilon_{n-1}}F_{\varepsilon_n}(x)) \tag{5.5}$$

where now (5.5) is written only for allowed $\varepsilon_1 \cdots \varepsilon_n$ strings, and $\sum'$ means that only allowed strings are summed. Next, we define

$$\psi_{\varepsilon_1\cdots\varepsilon_n}(x) = \psi(F_{\varepsilon_1}\cdots F_{\varepsilon_n})(x) \tag{5.6}$$

$$\lambda\psi_{\varepsilon_1\cdots\varepsilon_n}(x) = \sum_{\varepsilon_0}{}' |F_{\varepsilon_0}'(F_{\varepsilon_1}\cdots F_{\varepsilon_n}(x))|^\beta \psi_{\varepsilon_0\cdots\varepsilon_{n-1}}(F_{\varepsilon_n}(x)) \tag{5.7}$$

The form of (5.7) suggests a Markov diagrammatic representation. One draws nodes for each allowed string of n ε's and unidirectionally links one node into another that agrees in its first $n-1$ ε's with last $n-1$ ε's of the former. Figure 6 represents one equation of the system (5.7) and is self-explanatory. To use Fig. 6, the sum of a $\psi(F_{\varepsilon_n}(x))$ timmes an F' link factor for all links into $\varepsilon_1 \cdots \varepsilon_n$ produces λ times the $\psi(x)$ associated with the target node $\varepsilon_1 \cdots \varepsilon_n$. Only "legal" nodes are drawn and "legal" links connect them. For example, the system (5.4) for the tree of Fig. 5 is determined by the graph of Fig. 7.

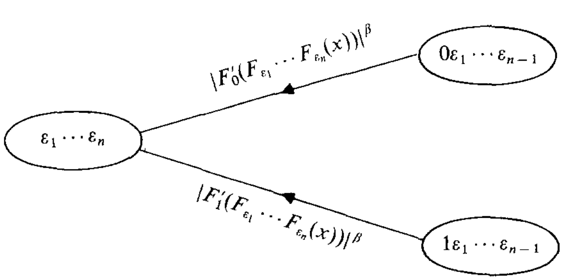

Fig. 6.

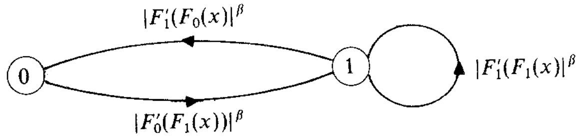

Fig. 7.

For $\beta = 0$ each link receives the factor $+1$; the largest λ is then the reciproval of the smallest positive zero of the determinant of the graph. By well-known methods, for Fig. 7 this determinant reads

$$0 = 1 - z - z^2 \to z_< = \frac{\sqrt{5}-1}{2} \to \lambda_> = z_>^{-1} = \frac{\sqrt{5}+1}{2}$$

Should the F's be linear, λ is again computed from the determinant of the graph, the links of which now are constants raised to the power β.

We thus see that the machinery of presentation functions easily extends to incomplete n-ary trees, and the thermodynamic information is readily available.

6. THE PERIOD DOUBLING DYNAMICS OF A PRESENTATION FUNCTION

I stated at the end of Section 2 that a dynamics can be associated with an *a priori* specified presentation function, in effect reversing the line of thought that led to presentation functions. By definition, this dynamics should generate the nth inverse images under E by the forward iteration of the dynamics. Since the dynamics will either possess orbits of length 2^n, one for each n, or a specified orbit of period 2^∞, the dynamics must be that of a period doubling fixed point. Let us now work out this connection. We shall see that a new method is afforded for determining these fixed points that perhaps can lead to a rigorous proof of their isolated existences.

In general there are 2^n nth inverses of E, labeled as $x_i^{(n)}$, $i = 0, \ldots, 2^n - 1$. By (2.6),

$$F_\varepsilon(x_i^{(n)}) = x_{2i+\varepsilon}^{(n+1)} \tag{6.1}$$

so that

$$F_1 F_0^{-1}(x_{2i}^{(n)}) = x_{2i+1}^{(n)} \tag{6.2}$$

Defining

$$g_0(x) = F_1 F_0^{-1} \tag{6.3}$$

g_0 provides the map that, applied to points of the right-hand piece of Fig. 1, images them to the central piece. Notice that g_0 images points at one level into others at the same level n.

Next observe that the range of F_1^r on the right-hand points is

$$F_1^r(x_{2i}^{(n)}) = x_{2^r(2i+1)-1}^{(x+r)} \tag{6.4}$$

Thus,

$$g_r(x) = F_0^r\, g_0 F_1^{-r}(x) \tag{6.5}$$

performs the mapping

$$g_r\colon \quad x_{2^r(2i+1)-1}^{(n)} \to x_{2^r(2i+1)}^{(n)} \tag{6.6}$$

so that the central (odd-indexed) points are mapped by the union of the restrictions of g, g_r, defined on intervals including just those x_i for which $i+1=2^r$ (odd). For this to make sense, the intersection of the interiors of the domains of any two distinct g_r must be empty. Moreover, the interval on which g_r is defined must include the relevant $x_i^{(n)}$ for *all n*. Since all x_{2i} lie within a fixed proper subinterval of the whole interval containing the support of E and since F_1 is a monotone contraction, this can be ascertained. In fact, provided E has a "gap" in its dommain of definition, as in Fig. 1, these intervals are totally disjoint; whereas, if there is no gap, as in Fig. 3, then intervals abut with empty interior intersections. Thus, the construction is as we have said. It should be noted that the domains of g_r as r diverges converge toward the fixed point of F_1, which, as we shall see, must then be the "critical point" of the map g.

By (6.3) and (6.6) we see that the range of g_0 includes the union of the domains of g_r; the range of each g_r is included within the domain of g_0. Notice by (6.3) that

$$g_{r-1} = F_0^{-1}\, g_r F_1 = F_1^{-1}\, g_0\, g_r F_1 \tag{6.7}$$

that is, g^2 restricted to the domain of g_r for any $r \geqslant 1$ is smoothly conjugated by the F_1^{-1} part of E to g. Similarly,

$$g_{r-1} = F_0^{-1} g_r\, g_0 F_0 \tag{6.8}$$

so that g^2 restricted to the domain of g_0 is smoothly conjugate by the F_0^{-1} part of E to g. Thus, given any E, it serves the conjugating role of (2.9) for the g constructed by (6.3) and (6.5).

So long as F_1 is a smooth, monotone contraction with fixed point at x_c we can smoothly conjugate it to the linear transformation determined by its derivative at its fixed point. Calling this derivative α^{-1}, one has

$$F_1'(x_c) = \alpha^{-1} \tag{6.9}$$

and (6.7) in these new coordinates reads

$$g_{r-1} = \alpha g_0 g_r \alpha^{-1} \tag{6.10}$$

so that we have the usual period doubling fixed-point equation

$$g = \alpha g^2 \alpha^{-1}$$

which opened Section 2.

By way of example, if we turn to Example 2 of Section 4 for the Farey tree, $F_1(x)$ in (4.8) can be conjugated to homogeneous linear form through the fractional linear conjugacy

$$\tilde{F}_1 = hF_1h^{-1} \tag{6.11}$$

with

$$h(x) = \frac{1-\rho^{-1}x}{1+\rho x}, \qquad h^{-1}(x) = \frac{1-x}{\rho^{-1}+\rho x} \tag{6.12}$$

yielding

$$\tilde{F}_1(x) = -\rho^2 x, \qquad \alpha^{-1} = -\rho^2 \tag{6.13}$$

and

$$\tilde{F}_0(x) = \frac{1+2\rho x}{2\rho^{-1}-x} \tag{6.14}$$

This "canonical" form of E is depicted in Fig. 8.

By (6.3),

$$g_0(x) = -\rho^2 \tilde{F}_0^{-1}(x) = \frac{\rho^2 - 2\rho x}{2\rho + x} \tag{6.15}$$

with the important property that $g_0^2(x) = x$, so that the fixed point of g_0 has eigenvalue -1. Then by (6.5) all the other g_r can be explicitly computed. This g is the Farey map we have discussed elsewhere (see footnote 5), and is exhibited in Fig. 9.

The reason for considering the Farey tree lies in the numerical fact that critical mode locking of oscillators has the property that the complement of the mode-locked intervals has the universal dimension of 0.87. A theory for this numerical result is still lacking. Figure 8 pertains to the kindred but simpler subcritical problem, for which the dimension is 1. A careful analysis of the subcritical problem reveals that the thermodynamics

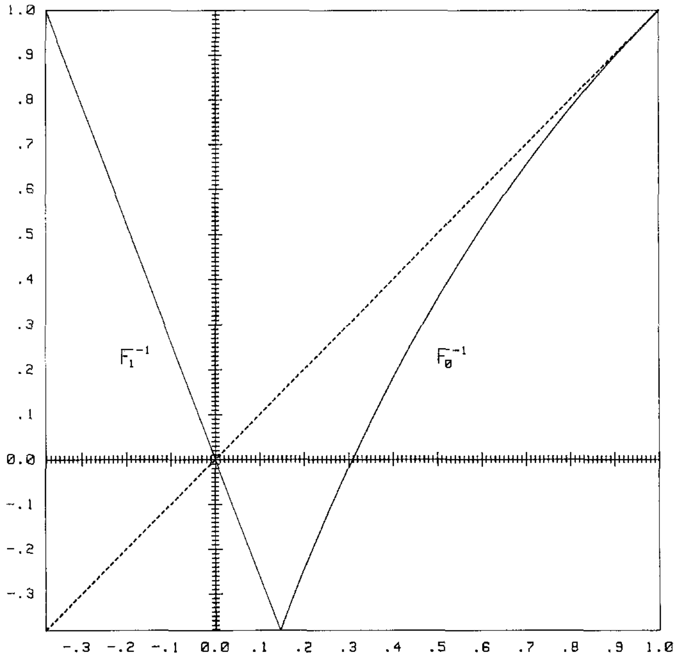

Fig. 8. The Farey presentation function of Fig. 3 after the conjugacy that brings F_1 to its canonical linear form, Eq. (6.13).

is saturated by the subtree at golden mean rotation. This part of the tree is universally determined by the unstable manifold of the renormalization-group fixed point of golden mean rotation. As we have just seen, for the subcritical case, h of (6.12) conjugates the entire tree to the golden mean subtree. However this works out for criticality, if the result is universal, it must also pertain to the unstable manifold of the critical golden mean fixed point. Thus there is an $\tilde{F}_1$ which is linear, but the derivative in criticality is the renormalized δ

$$\delta_r = -2.83361... \tag{6.16}$$

instead of the subcritical $\delta = -\rho^2 = -2.618...$. Thus, the arc $\tilde{F}_1$ for the critical problem is exactly known to be

$$F_1(x) = \delta_r^{-1} x$$

The corresponding F_0 is still not available. It suffices to say that it is a numerical fact that the critical case is smooth to within 10^{-8} (available precision) and is very well fit by

$$F_0(x) \approx x + k(1-x)^\nu, \qquad \nu \approx 1.37, \quad k \approx 0.408 \tag{6.17}$$

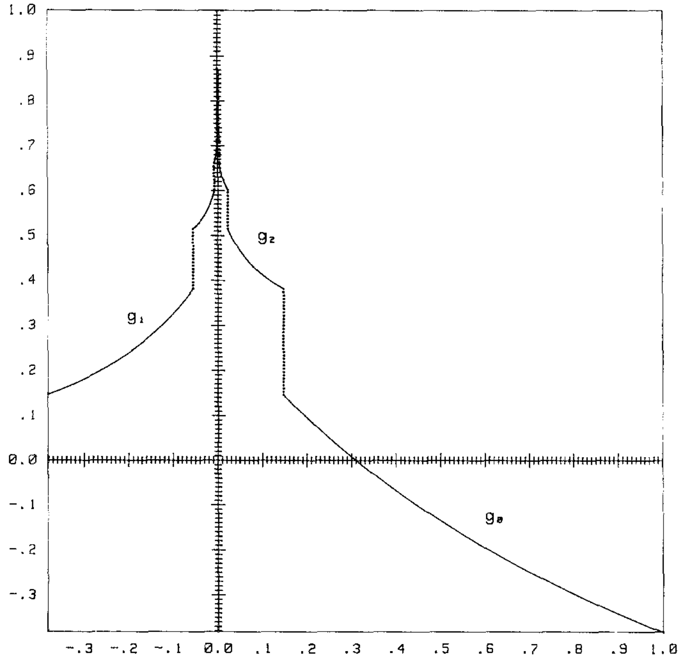

Fig. 9. The "Farey map" period doubling fixed point, the forward dynamics of which is the Farey tree.

and E is depicted in Fig. 10. The opening of the gap in E of course implies a dimension smaller than 1; the fit of (6.16) (theoretical) and (6.17) indeed reproduces the numerical 0.87 when entered in (3.15).

Let us end this section with the determination of g for the approximation (4.1) and (4.2) to the E of Fig. 1 for quadratic period doubling. We shall discover from this inquiry a general method for determing solutions to renormalization-group fixed-point equations.

We take

$$F_1 = \alpha^{-1} x \tag{6.18}$$

$$F_0 = 1 - \sigma_0(1 - x) \tag{6.19}$$

where (6.18) for the correct F_0 is exact. Notice by (6.5) that

$$g_r = 1 - \sigma_0^r[1 - g_0(\alpha^r x)] \tag{6.20}$$

where by (6.3)

$$g_0 = \alpha^{-1}[1 - \sigma_0^{-1}(1 - x)] \tag{6.21}$$

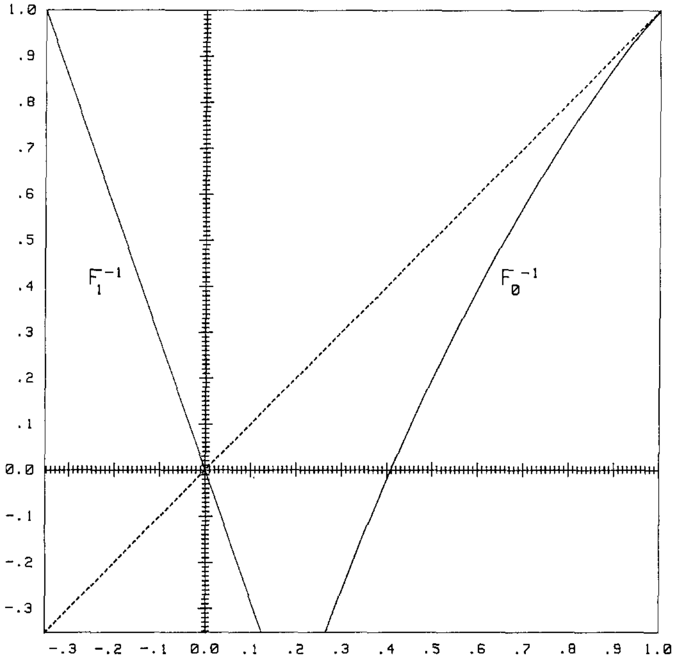

Fig. 10. An excellent *numerical* fit to F_0 of Eq. (6.17) for cubically critical mode locking. This presentation function, iterated, determines mode locking intervals and the well-known $\beta_H = 0.87$ to high accuracy. It is to be noted that since ν differs from 1.5, the intermittency argument Q^{-3} is misleadingly unimportant for β_H.

The requirement we impose on $\{g_r\}$ is that they should be the restrictions of some smooth function with a quadratic maximum at $x = 0$ [which by (6.20) has the critical value of 1]. Notice by (6.20) that

$$\begin{aligned} g_r(\alpha^{-r}\hat{x}) &= 1 - [1 - g_0(\hat{x})]\sigma_0^r \\ &= 1 - \frac{1 - g_0(\hat{x})}{\hat{x}^2}\left(\frac{\sigma_0}{\alpha^{-2}}\right)^r (\alpha^{-r}\hat{x})^2 \end{aligned} \tag{6.22}$$

Thus, for any $\hat{x}$, if we fix

$$\sigma_0 = \alpha^{-2} \tag{6.23}$$

then (6.22) is compatible with

$$g = 1 - \mu x^2 \tag{6.24}$$

for

$$\mu = \frac{1 - g_0(\hat{x})}{\hat{x}^2} \tag{6.25}$$

Since by (6.21) g_0 entails the unknown constant α^{-1}, demanding that μ satisfies (6.25), at two distinct values of $\hat{x}$ will determine α^{-1}. However, choosing F_0 (as we have) to be the linear arc of (6.19), should the two choices of $\hat{x}$ be x_0 and x_2 of Fig. 1 [so that $x_0=1$, $x_2=F_0(\alpha^{-1})$], then the linear arcs of g_r of (6.20) will be a linear polygonal period doubling fixed point with inscribing endpoints lying on the parabola (6.24) and so possessing an asymptotic quadratic critical point. Thus, set in (6.25)

$$\mu=\frac{1-g_0(1)}{1}=1-\alpha^{-1} \tag{6.26a}$$

and

$$\mu=\frac{1-g_0(x_2)}{x_2^2}=\frac{1-g_0[F_0(\alpha^{-1})]}{x_2^2}=\frac{1-\alpha^{-2}}{[1-\alpha^{-2}(1-\alpha^{-1})]^2} \tag{6.26b}$$

Eliminating μ in (6.26a), (6.26b) results in a quintic for α^{-1} with the root

$$\alpha = -2.48634\ldots$$

with an error of 0.66% from the actual result of $-2.502907875\ldots$. The polygonal fixed point is depicted in Fig. 11.

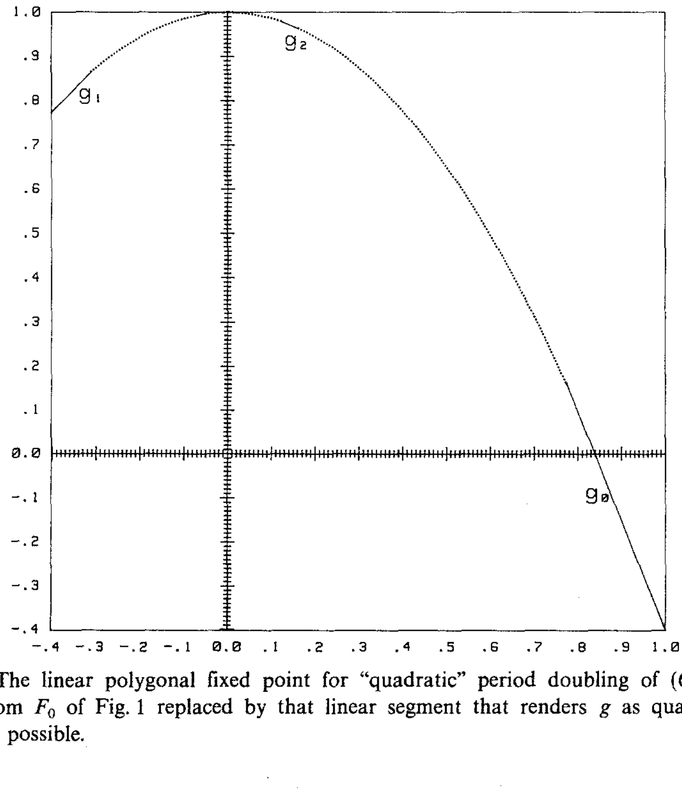

Fig. 11. The linear polygonal fixed point for "quadratic" period doubling of (6.22) that follows from F_0 of Fig. 1 replaced by that linear segment that renders g as quadratically smooth as possible.

Thus, in review, by requesting the $\{g_r\}$ (in some sense) to be the restrictions of a smooth, quadratically critical map, I have been led to an excellent linear polygonal solution to the fixed-point equation of the period doubling renormalization group. I now want to elevate this calulation to a systematically exact one. In order to do so, I must first explore the nature of approximations of F_0 by successively more linear segments, which I shall face in the next section through the idea of the scaling function. Having done that, in the final two sections I produce the general calculation for the fixed point and discover that to all orders (beyond the lowest, as a numerical observation) the solution is obtained as a globally stable fixed point of a natural dynamics in the space of all scalings. I shall mention my grander thoughts of the meaning of this dynamics in a final afterword.

To fortify the reader, Section 8 is technically arduous, with a rather simple final result. In erecting equations directly for the scalings, and moreover in discovering a natural and exceptionally well behaved flow on them, I believe new ground in being charted and, so beg the reader's indulgence.

7. THE RELATIONSHIP BETWEEN PRESENTATIONS AND SCALING FUNCTIONS

I presented a Markov graphical method of successive approximation to dynamical thrmodynamics in ref. 4. The links on those graphs were *constants* determined by the trajectory scaling function. In Section 5 (e.g., Fig. 7) I have drawn very low-order *exact* Markov graphs utilizing presentation functions. Should one proceed to construct successive higher-order duals of the F graphs, the links would have (F')'s of successively restricted arguments on them. With F_ε of bounded nonlinearity, these would approach constants, and so, obviously, would be identifiable with (constant) values of the scaling function σ. I now precisely work out that connection. In particular, it shall be possible to understand how an F_0 of a finite number of linear arcs is an expression of a σ of a finite number of constant values. Thus, from any smooth nonlinear F_ε one directly infers, in "modern" parlance, a multifractal of infinite scaling complexity. The ease with which one can deduce, for example, the thermodynamics of these objects should convice the reader that the machinery I have been erecting and discussing in these pages is very powerful.

The idea of scaling functions[6] is very simple. It is constituted of the observation that the quotient of small distances (asymptotically infinitesimals) is obviously invariant under smooth coordinate transfor-

[6] The scaling function is invented in ref. 8; see also refs. 9 and 10.

mations, and hence under the dynamics itself. That is, scaling properties are well ordered in *dynamical order* but not in space.

Using the ε notation of (2.6), define the scaling at a point on the orbit by

$$\sigma(\varepsilon_m \cdots \varepsilon_0) = \frac{x_{0\varepsilon_0 \cdots \varepsilon_0} - x_{1\varepsilon_m \cdots \varepsilon_0}}{x_{\varepsilon_m \varepsilon_{m-1} \cdots 0} - x_{\bar{\varepsilon}_m \varepsilon_{m-1} \cdots \varepsilon_0}} \tag{7.1}$$

where we are reexpressing the idea behind (3.3) that determines the smallest distance that can be identified with $m+1$ or m epsilons, and forming their quotient ($\bar{\varepsilon}$ denotes the complement of ε). Notice that the first of the numerator and denominator terms are evaluated at identical dynamical indexes, ensuring the invariance of σ under smooth coordinate transformations. (σ is an invariant, and generally a fuller one than the set of all periodic orbit eigenvalues,[7] the literature notwithstanding. Indeed, one cannot simply follow the Markov idea of the introduction to this section simply because thermodynamics and asymptotic exponential quantities are degenerate over the fuller information in σ.) Employing (2.6), one has that (7.1) relates σ to F:

$$\sigma(\varepsilon_m \cdots \varepsilon_0) = \frac{F_{\varepsilon_0} \cdots F_{\varepsilon_m} F_0(x_0) - F_{\varepsilon_0} \cdots F_{\varepsilon_m} F_1(x_0)}{F_{\varepsilon_0} \cdots F_{\varepsilon_m}(x_0) - F_{\varepsilon_0} \cdots F_{\bar{\varepsilon}_m}(x_0)} \tag{7.2}$$

The details to follow will implement the intuitive idea that after enough F's have been aplied to x_0, whether the first F is F_0, F_1, F_{ε_m}, or $F_{\bar{\varepsilon}_m}$, the resulting x's are all contracted enough toward one another as to lie within a single domain over which F is linear, so that the remaining lower-indexed F_ε's produce identical slope factors in numerator and denominator. That is, with F_0 a finite number of linear restrictions, there is only a finite number of distinct values of σ no matter how large m is: σ depends only upon a certain number of the highest indexed ε's. Let us see how this works out.

We begin by defining $F_0^{(n)}$ to be linear over each of the 2^n disjoint intervals of its domain. Accordingly, set

$$(F_{0,k}^{(n)})^{-1} \quad \text{linear on } (2k, 2k+2^{n+1}), \qquad k = 0,..., 2^n - 1 \tag{7.3}$$

$$F_{0,k}^{(n)}: \quad x_{l \cdot 2^n + k} \to x_{2(l \cdot 2^n + k)} \tag{7.4}$$

$$F_0^{(n)}(x_m) = F_{0, m \bmod 2^n}^{(n)}(x_m) \tag{7.5}$$

Notice in (7.3) that we use the dynamics to partition the domain of F^{-1} to be intervals whose endpoints are closest return neighbors. [The *order* of the endpoints might be reversed to $(2k+2^{n+1}, 2k)$ depending upon k.]

[7] Unpublished work of Dennis Sullivan, presented at the 1987 Noto Summer School, in which he establishes the scaling function as the maximal invariant for $C^{1+\alpha}$ geometries.

Condition (7.3) by (6.3) now determines restrictions of g_0:

$$g_{0,k}^{(n)} = F_1(F_{0,k}^{(n)})^{-1}: \qquad x_{2(k+l\cdot 2^n)} \to x_{2(k+l\cdot 2^n)+1} \tag{7.6}$$

and similarly by (6.5) for g_r:

$$g_{r,k}^{(n)} = F_0^r\, g_{0,k}^{(n)} F_1^{-r} \tag{7.7}$$

defined on the domain $F_1^r(\text{dom}\, g_0)$ so that

$$g_{r,k}^{(n)}: \quad x_{2^r(2k+1+l\cdot 2^{n+1})-1} \to x_{2^r(2k+1+l\cdot 2^{n+1})} \tag{7.8}$$

The definition of (7.7) is completed with the understanding that each F_0 in it is to be that $F_{0,k}^{(n)}$ appropriate to the domain as given by (7.5). I want to explicitly record this. Since $\text{dom}(g_r) = F_1^r(\text{dom}\, g_0) = \alpha^{-r}(\text{dom}\, g_0)$, we have

$$g_{r,k}^{(n)}(\alpha^{-r} x_{2k+\varepsilon 2^{n+1}}) = F_0^r\, g_{0,k}^{(n)}(x_{2k+\varepsilon 2^{n+1}}) = F_0^r(x_{2k+1+\varepsilon 2^{n+1}})$$

$$\overset{(r \geqslant n)}{=} [F_{0,0}^{(n)}]^{r-n} F_{0,2^{n-1}(2k+1)\,\text{mod}\, 2^n}^{(n)} \cdots F_{0,2(2k+1)\,\text{mod}\, 2^n}^{(n)} F_{0,2k+1}^{(n)}(x_{2k+1+\varepsilon 2^{n+1}}) \tag{7.9}$$

The most important aspect of (7.9) for the extension of the calculation of σ_1 at the end of Section 6 is that, apart from n other factors, g_r utilizes the one restriction $F_{0,0}$ that includes the fixed point of F_0 at $x = x_0 = 1$. I shall pick this up in the next section. For the moment, let us return to (7.2) now that the F's have been explicitly defined.

Note that the rightmost n of the F's of each denominator term in (7.2) produce

$$F_{\varepsilon_{m-n}}^{(n)} \cdots F_{\varepsilon_{m-1}}^{(n)} F_{\varepsilon}^{(n)}(x_0) = x_{\varepsilon 2^n + \varepsilon_{m-1} \cdots \varepsilon_{m-n}} = x_{\varepsilon 2^n + k} \tag{7.10}$$

for some $k < 2^n$. By (7.5) we see that (7.10) is in the domain of the *same* $F_{0,k}^{(n)}$ independent of ε. Identically, so, too, are the numerator terms

$$F_{\varepsilon_{m-n}}^{(n)} \cdots F_{\varepsilon_m}^{(n)} F_{\varepsilon}^{(n)}(x_0) \tag{7.11}$$

It follows further that the rest of the leftmost F's (of lower-indexed ε's) preserve the fact that each numerator and denominator x persists to lie in the domain of the same F restriction. Since each F is *linear*, we have

$$\begin{aligned} &F_{\varepsilon_0}^{(n)} \cdots F_{\varepsilon_{m-1}}^{(n)} F_{\varepsilon_m}^{(n)}(x_0) - F_{\varepsilon_0}^{(n)} \cdots F_{\varepsilon_{m-1}}^{(n)} F_{\bar{\varepsilon}_m}^{(n)}(x_0) \\ &\quad = F_{\varepsilon_0}^{(n)\prime} \cdots F_{\varepsilon_{m-n-1}}^{(n)\prime}(x_{\varepsilon_m 2^n + k} - x_{\bar{\varepsilon}_m 2^n + k}) \end{aligned}$$

and

$$\begin{aligned} &F_{\varepsilon_0}^{(n)} \cdots F_{\varepsilon_m}^{(n)} F_0^{(n)}(x_0) - F_{\varepsilon_0}^{(n)} \cdots F_{\varepsilon_m}^{(n)} F_1^{(n)}(x_0) \\ &\quad = F_{\varepsilon_0}^{(n)\prime} \cdots F_{\varepsilon_{m-n-1}}^{(n)\prime}(x_{\varepsilon_m 2^n + k} - x_{\varepsilon_m 2^n + k + 2^{n+1}}) \end{aligned}$$

Performing the quotient of (7.2), we find that the derivatives cancel, and we have

$$\sigma(\varepsilon_m \cdots \varepsilon_0) = \sigma(\varepsilon_m \cdots \varepsilon_{m-n}) = \frac{x_k - x_{k+2^{n+1}}}{x_k - x_{k+2^n \bmod 2^{n+1}}} = \sigma_n(k) \qquad (7.12)$$

where

$$k = 2^n\varepsilon_m + \cdots + 2^0\varepsilon_{m-n} = 0,..., 2^{n+1} - 1 \qquad (7.13)$$

Thus, with $F_0^{(n)}$ linearly defined on 2^n intervals, the set of all $\sigma(x_m)$ possesses 2^{n+1} distinct values constant over all but the highest indexed $n+1$ ε's. Also, 2^{n+1} independent parameters completely determine the 2^n linear functions $F_{0,k}^{(n)}$ (and *not* just the 2^n slopes $F_{0,k}^{(n)\prime}$), and, as we shall soon see, constitutes a most natural parameterization for dynamical purposes.

8. THE SCALING FUNCTION THEORY OF THE PERIOD DOUBLING FIXED POINT

Let us review what has been done so far. I started with the usual period doubling fixed point g and realized that its dynamics could also be determined by the backward dynamics (inverses) of an expanding map E, the presentation function. I next observed that the inverses of E, the functions F_ε, most naturally allow the determination of the thermodynamics of the dynamics of g. I next commented that forgetting g, the scheme of the F_ε is generally applicable to objects defined on trees. I then inverted the exposition, showing that the F_ε for any tree still determine the identical dynamics of some generalized period doubling fixed point that can be explicitly constructed from the F's. Moreover, it was shown at the end of Section 6 that requiring that fixed point to be smooth is a sufficient principle to determine the F_ε that give rise to it. In the last section it was realized that polygonal linear F's are equivalent to piecewise constant valued scaling functions, the latter a rich invariant under coordinate transformations. I am now prepared to again invert the exposition and discover how to frame the discussion of the underlying dynamics purely in the language of the scalings. I shall proceed to deduce the general equations that the scaling function satisfies, and then show in Section 9 that these equation can be solved by erecting a natural dynamics on the space of scalings which possesses as a globally stable fixed point the scaling function, hence the presentation function, and hence the period doubling fixed point that expresses the underlying dynamics.

Return to (7.9), and notice, as pointed out there, that the index r appears only as the power of the one restriction $F_{0,0}$. By (7.4), x_0 is its fixed point, so that with $x_0 = 1$ and $F'_{0,0} = \sigma_0$, we have

$$F_{0,0}^{(n)}(x) = 1 - \sigma_0(1 - x) \tag{8.1}$$

yielding

$$[F_{0,0}^{(0)}]^r(x) = 1 - \sigma_0^r(1 - x) \tag{8.2}$$

We can thus rewrite (7.9) explicitly in its r dependence for all $r \geqslant n$ as

$$g_{r,k}^{(n)}(\alpha^{-r} x_{2k+\varepsilon \cdot 2^{n+1}}) = 1 - \sigma_0^r U_{k,\varepsilon}^{(n)} \tag{8.3}$$

where

$$U_{k,\varepsilon}^{(n)} = \sigma_0^{-n}[1 - F_{0,2^n-1}^{(n)} \cdots F_{0,2(2k+1) \bmod 2^n}^{(n)} F_{0,2k+1}^{(n)}(x_{2k+1+\varepsilon 2^{n+1}})] \tag{8.4}$$

Now, just as we observed in(6.23), if we take

$$\sigma_0 = \alpha^2 \tag{8.5}$$

then (8.3) reads

$$g_{r,k}^{(n)}(\alpha^{-r} x_{2k-\varepsilon \cdot 2^{n+1}}) = 1 - (\alpha^{-r} x_{2k+\varepsilon \cdot 2^{n+1}})^2 \frac{U_{k,\varepsilon}^{(n)}}{x_{2k+\varepsilon \cdot 2^{n+1}}^2} \tag{8.6}$$

We now see, in analogy to (6.24)–(6.26), that by *requiring* that

$$U_{k,\varepsilon}^{(n)} = \mu_n x_{2k+\varepsilon \cdot 2^{n+1}}^2 \qquad \text{for some } \mu_n \tag{8.7}$$

we shall achieve

$$g_{r,k}^{(n)}(x) \text{ “=” } 1 - \mu_n x^2 \qquad (r \geqslant n, \text{ all } k) \tag{8.8}$$

where the quotes about the equals sign mean that (8.8) is satisfied just at the endpoints of the domains of definition of $g_{r,k}^{(n)}$. Notice that (8.8) is imposed only for $r \geqslant n$. Thus, the calculation of order n requires that all *but* the lowest 2^n restrictions of g lie on a parabola. As n increases, these lowest-order pieces determine the actual fixed point g, and not a parabola. However, μ_n converges to $1/2\, g''(0)$. Let us make sure that we understand what is to be determined.

The points

$$x_{2k+\varepsilon \cdot 2^{n+1}}, \qquad \varepsilon = 0, 1; \quad k = 0, \ldots, 2^n - 1 \tag{8.9}$$

are, by (7.3) the endpoints of the range of $F_{0,k}^{(n)}$. Using (8.4) in (8.7) should result in a sufficient number of equations to determine the x's of (8.9). However, as each $F_{0,k}^{(n)}$ is linear, and by (7.4) obeys

$$(F_{0,k}^{(n)})^{-1}: \quad x_{2k+\varepsilon\cdot 2^{n+1}} \to x_{k+\varepsilon\cdot 2^n} \tag{8.10}$$

while

$$x_{2k+1} = F_1(x_k) = \alpha^{-1} x_k$$

the $F_{0,k'}^{(n)}$ *can* be parametrized by the x's of (8.9) *together* with the value of α^{-1}. Since, however, we have chosen the scale $x_0 = 1$, we still have between α^{-1} and the x_{2k} the required 2^{n+1} quantities to be determined by the system (8.7). More correctly, there is still the extra quantity μ_n of (8.7). However, the requirement that g possesses a *quadratic* critical point expressed by (8.5) provides the missing equation. We thus conjecture that the system of (8.7), (8.5), and $x_0 = 1$ possesses an isolated solution for the positive quantities x_{2k} of (8.9).

The reader should feel that this calculation is far from transparent. So it appears at this stage. However, recall that there are also precisely 2^{n+1} values $\sigma_n(k)$ of (7.12), one of which is $\sigma_n(0) = \sigma_0$ of (8.5). I promised that σ was a *natural* parametrization of the $F_{0,k}^{(n)}$. I shall now show why, and reduce the computation to one of systematic ease.

First, by (7.12),

$$\sigma_n(2^p(2k+1)) = \frac{x_{2^p(2k+1)} - x_{2^p(2k+1)+2^{n+1}}}{x_{2^p(2k+1)} - x_{2^p(2k+1)-2^n \bmod 2^{n+1}}} \quad \text{for} \quad 2k+1 < 2^{n+1-p} \tag{8.11}$$

However, as also expressed by (7.12),

$$\sigma_n[2^p(2k+1)] = \sigma[2^{r+p}(2k+1)] \qquad \text{all} \quad r \geqslant 0 \tag{8.12}$$

which by (7.1) is again a quotient of differences of x's, but now for sufficiently large r ($\geqslant n-p$) the image (the index is even) of an endpoint of the domain of a $g_r^{(n)}$ obeying (8.8); that is,

$$x^{2^r+p(2k+1)} = g_r^{(n)}(x_{2^r+p(2k+1)-1}) = 1 - \mu_n x^2_{2^r+p(2k+1)-1} \tag{8.13}$$

By (8.12), (8.11) is now the quotient of the diferences of the x^2's of (8.13). However, by (6.4) and (6.18),

$$x^2_{2^r+p(2k+1)-1} = \alpha^{-2r} x^2_{2^p(2k+1)-1} \tag{8.14}$$

so that (8.11) is the quotient of the differences of the squares in (8.14). Finally, denoting $2^p(2k+1)-1=l$, we have

$$\sigma_n(l+1)=\frac{x_l^2-x_{l+2^{n+1}}^2}{x_l^2-x_{l+2^n \bmod 2^{n+1}}^2} \tag{8.15}$$

to be compared with the defining formula

$$\sigma_n(l)=\frac{x_l-x_{l+2^{n+1}}}{x_l-x_{l+2^n \bmod 2^{n+1}}}, \qquad l=0,\ldots,2^{n+1}-1 \tag{8.16}$$

There is one important proviso on (8.15): its range of applicability is

$$l+1=1,\ldots,2^n-1,2^{n+1},\ldots,2^{n+1}-1 \quad \text{in} \quad (8.15) \tag{8.17}$$

This follows from the derivation of (8.15); should $p=n$ in (8.11), then k of (8.11) must equal zero. But then the second denominator index in that formula is $2^n+2^n \bmod 2^{n+1}=0$; however, x_0 is *never* the image of an x_m under $g_r^{(n)}$, whence the restriction in l of (8.17). Apart from the exceptions $l+1=0$ or 2^n (which I shall treat immediately), the content of (8.15) is that *the one-index advanced σ bears the identical relationship to the x^2's as does the σ of the unadvanced index to the same, but not squared, x's.* That is, the full content of the system (8.4), (8.7) is precisely this twice-defined encoding of the σ's.

To finish the deduction of the equations, let us turn to (8.16) for $l=2^n$:

$$\begin{aligned}\sigma_n(2^n)&=\frac{x_{2^n}-x_{3\cdot 2^n}}{x_{2^n}-x_0}\\ &=\frac{g_{n,0}^{(n)}(x_{2^n-1})-g_{n,1}^{(n)}(x_{3\cdot 2^n-1})}{g_{n,0}^{(n)}(x_{2^n-1})-1}\\ &=\frac{x_{2^n-1}^2-x_{3\cdot 2^n-1}^2}{x_{2^n-1}^2} \qquad [\text{by } (8.8)]\\ &=1-x_2^2 \qquad [\text{by } (6.4) \text{ and } (6.18)]\end{aligned}$$

That is,

$$1-x_2^2=\sigma_n(2^n) \tag{8.18}$$

Also, by (8.16) using (6.4) and (6.18),

$$\sigma_n(2^n-1)=\frac{x_{2^n-1}-x_{3\cdot 2^n-1}}{x_{2^n-1}-x_{2^{n+1}-1}}=\frac{1-x_2}{1-\alpha^{-1}} \tag{8.19}$$

However, with $l = 2^{n+1} - 1$ in (8.16), again using (6.4) and (6.18), we have

$$\sigma_n(2^{n+1} - 1) = \frac{x_{2^{n+1}-1} - x_{2^{n+2}-1}}{x_{2^{n+1}-1} - x_{2^n-1}} = -\alpha^{-1} \tag{8.20}$$

Thus, α^{-1} is also one of the σ values (the most obvious one) and, with (8.19), we have

$$1 - x_2 = \sigma_n(2^n - 1)[1 + \sigma_n(2^{n+1} - 1)] \tag{8.21}$$

Finally, noting (8.5) and (8.20), we can explicitly eliminate α^{-1} in

$$\sigma_n(0) = [\sigma_n(2^{n+1} - 1)]^2 \tag{8.22}$$

and write down the system of equations that determines σ:

$$\begin{aligned} 1 - [1 + \sigma_n(2^{n+1} - 1)]\, f_k^{(n)}[\sigma_n(i)] \\ = \{1 - f_k^{(n)}[\sigma_n(i+1)]\}^{1/2}, \qquad k = 1,..., 2^{n+1} - 1 \end{aligned} \tag{8.23}$$

where

$$f_k^{(n)}[\sigma_n(i+1)] = 1 - x_{2k}^2 \tag{8.24}$$

meaning that $f_k^{(n)}$ is that function of the set of one-index advanced σ's that is obtained by solving the set (8.15) for $1 - x_{2k}^2$ in terms of $1 - x_2^2$, which by (8.18) is simply $\sigma_n(2^n)$. The $f_k^{(n)}[\sigma_n(i)]$ on the left-hand side side of (8.23) is the same functional form as determined from (8.24) with each $\sigma_n(i+1)$ argument replaced by the argument $\sigma_n(i)$. The positive square root taken in (8.23) reflects the fact that $x_{2k} > 0$ in (8.24). Before going further, let us work out the two lowest orders, $n = 0, 1$, of this theory.

At $n = 0$, there are two values of σ, σ_0 and σ_1. With $k = 1$ in (8.24),

$$1 - x_2^2 = \sigma_1 = f_1(\sigma(i+1))$$

and (8.23) reads

$$1 - (1 + \sigma_1)\sigma_0 = (1 - \sigma_1)^{1/2}$$

which together with (8.22), $\sigma_0 = \sigma_1^2$, is easily seen to be the earlier result (6.26).

At order $n = 1$, we can see the machinery generally at work. By (8.22),

$$\sigma_0 = \sigma_3^2 \tag{8.25}$$

By (8.24),

$$k=1:\quad f_1(\sigma_{i+1})=1-x_2^2=\sigma_2 \qquad \text{[Eq. (8.18)]} \tag{8.26}$$

$$k=2:\quad f_2(\sigma_{i+1})=1-x_4^2=\frac{x_0^2-x_4^2}{x_0^2-x_2^2}(1-x_2^2)=\sigma_1\sigma_2 \tag{8.27}$$

$$k=3:\quad f_3(\sigma_{i+1})=1-x_6^2=(1-x_2^2)+\frac{x_2^2-x_6^2}{x_2^2-x_0^2}(x_2^2-x_0^2)$$

$$=\sigma_2(1-\sigma_3). \tag{8.28}$$

Entering these results in (8.23), we have

$$k=1:\quad 1-(1+\sigma_3)\sigma_1=(1-\sigma_2)^{1/2} \tag{8.29}$$

$$k=2:\quad 1-(1+\sigma_3)\sigma_0\sigma_1=(1-\sigma_1\sigma_2)^{1/2} \tag{8.30}$$

$$k=3:\quad 1-(1+\sigma_3)\sigma_1(1-\sigma_2)=[1-\sigma_2(1-\sigma_3)]^{1/2} \tag{8.31}$$

Together with (8.25), (8.29)–(8.31) determine a unique solution for $\sigma_0 \cdots \sigma_3 \in (0, 1)$ given numerically by

$$\sigma_0=0.1573393...$$

$$\sigma_1=0.1759069...$$

$$\sigma_2=0.4310046...$$

$$\sigma_3=0.3966602...$$

with $|\alpha|=\sigma_3^{-1}=2.521049...$, an error similar to that of the $n=0$ solution of (6.26). As we proceed to higher n, the solutions geometrically converge toward the actual result.

There are several comments to make at this point. The first is that the extraction of $1-x_{2k}^2$ for each k from (8.15) is a straightforward task, so that any equation in any order of approximation can immediately be written down. However, as the solution to the full set of equations for a given order n becomes arduous and numerically problematic, I leave this exercise to the reader. Second, there is an important systematics that determines half of the order $n+1$ equations from all those of order n, so that the solution to order n, doubled up by pairs, exactly satisfies the first half of the order $n+1$ equations. By the bounded nonlinearity of F_0 follows the exponentially weak dependence of σ on lower-index ε's. This means that the remaining half to the order $n+1$ equations fail to be satisfied with exponentially small error for large n. Thus, there is good theoretical reason to believe that successive orders of approximation define a convergent theory. These results follow from the exact linearity of F_1:

$$\sigma_{n+1}(2k+1) = \frac{F_1(x_k) - F_1(x_{k+2^{n+1}})}{F_1(x_k) - F_1(x_{k+2_n \bmod 2^{n+1}})}$$

$$= \frac{x_k - x_{k+2^{n+1}}}{x_k - x_{k+2^n \bmod 2^{n+1}}} = \sigma_n(k)$$

It is then easy to see that by also replacing $\sigma_{n+1}(2k)$ by $\sigma_n(k)$, the first 2^{n+1} of Eqs. (8.23) at order $n+1$ becomes precisely the full set at order n.

However, as I have already alluded to, having written down the 2^{n+1} equations of order n in no way implies that we are close to a solution. Indeed, even using the trial approximation (the doubling up by pairs mentioned above)

$$\sigma_{n+1}(2k+1) = \sigma_{n+1}(2k) = \sigma_n(k)$$

while precisely satisfying half the order $n+1$ equations, and failing to satisfy the second half with exponentially decreasing errors, still fails to fall with the basin of attraction of Newton's method to the order $n+1$ solution. However, we can do much better.

9. DYNAMICS IN THE SPACE OF SCALINGS

I showed in the last section that the scalings σ can be written as a quotient of differences of either coordinates, or in critical image, the squares of coordinates. (I used this idea in the past to determine the discontinuities of σ, which, indeed, are exponentially graded accoding to closeness of approach to the critical point of g.) The formulas that fix σ are precisely the requirements that both evaluations agree.

Inspecting (8.15), one sees that

$$\sigma_n(l+1) = \frac{x_l^2 - x_{l+2^{n+1}}^2}{x_l^2 - x_{l+2^n \bmod 2^{n+1}}^2} = \sigma_n(l) \frac{x_l + x_{l+2^{2+1}}}{x_l + x_{l+2^n \bmod 2^{n+1}}} \tag{9.1}$$

or

$$\frac{\sigma_n(l+1)}{\sigma_n(l)} - 1 = \frac{x_{l+2^{n+1}} - x_{l+2^n \bmod 2^{n+1}}}{x_l + x_{l+2^n \bmod 2^{n+1}}} \sim \sigma_{\text{eff}}^n \tag{9.2}$$

so long as $|x_l| \sim 1$. Thus, provided that

$$l \bmod 2^r \neq 0 \qquad \text{for} \quad r = O(n) \tag{9.3}$$

$\sigma_n(l+1)$ is exponentially (in n) close to $\sigma_n(l)$. This suggests a simple strategy to (recursively) enfore the agreement of (8.15) and (8.16): Assume a given set of σ's and update them (by a dynamics in the 2^{n+1}-dimensional space of σ_n) by (i) inverting (8.15) to determine the x_{2k}^2 in terms of the $\sigma_n(i)$; (ii) taking square roots to produce the x_{2k}; and (iii) using the x_{2k} in (8.16) to determine new $\sigma_n'(i)$. That is, we have a transformation T in the space of scaling functions

$$\sigma_n' = T_n[\sigma_n] \tag{9.4}$$

which we now hope relaxes to the common solution of (8.15) and (8.16). For l's not "too" dyadically small [i.e., (9.3)], we anticipate from the discussion of (9.1) and (9.2) that

$$\sigma_n'(l-1) \approx \sigma_n(l) \tag{9.5}$$

Thus, the great bulk of the dynamics of T is the simple difference delay dynamics of (9.5) with more serious right-hand sides at the dyadically smallest values of l (e.g., 2^n). This implies in order n with 2^{n+1} σ's, after a major discontinuity, another 2^{n+1} steps are required for this significant modification to propagate "around" the set of σ's in order to be transformed again. Thus, if we denote by λ_n the eigenvalue of the T_n convergence to the stable fixed point σ_n^*, then we expect

$$\lambda_n^{2^{n+1}} \sim \Lambda \tag{9.6}$$

Indeed, after minor details are fixed [as before, $l+1 \neq 2^n$ in (8.15)], this program works perfectly. Thus, as I determine numerically, (9.6) is correct with

$$\Lambda \approx 0.70 e^{\pm 0.23 i} \tag{9.7}$$

However, not only does T possess σ^* as a stable fixed point, but so far I can determine numerically (although my search has not been exhaustive), T also possesses σ^* as a *global* attractor with basin $\{(x_1, ..., x_{2^{n+1}}) \in (0, 1)^{2^{n+1}}\}$. One can verify this in the lowest-order models where for all but 2^n sigmas, all other σ_N exactly satisfy the delay dynamics of (9.5). To fix ideas and to explain part of the last comment, let us now work out the $n=0, 1$ scaling dynamics using (9.5) taken with identity in I^N, where I is (0, 1).

The first step is a simple piece of systematics. Return to (8.15) and evaluate

$$\sigma_n[2^{n-r}(2i+1)] = \frac{x_{2^{n-r}(2i+1)-1}^2 - x_{2^{n-r}(2i+1)+2^{n+1}-1}^2}{x_{2^{n-r}(2i+1)-1}^2 - x_{2^{n-r}(2i+1)+2^n-1 \bmod 2^{n+1}}^2}$$

$$n \geq r \geq 1, \qquad 2i < 2^{r+1} \tag{9.8}$$

Using (6.4) and (6.18) as in (8.14), we obtain

$$\sigma_n[2^{n-r}(2i+1)] = \frac{x_{2i}^2 - x_{2i+2^{r+1}}^2}{x_{2i}^2 - x_{2i+2^r \bmod 2^{r+1}}^2}, \qquad n \geqslant r \geqslant 1, \quad 2i < 2^{r+1} \tag{9.9}$$

and the related result

$$\sigma_n[2^{n-r}(2i+1) - 1] = \frac{x_{2i} - x_{2i+2^{r+1}}}{x_{2i} - x_{2i+2^r \bmod 2^{r+1}}}, \qquad n \geqslant r \geqslant 0, \quad 2i < 2^{r+1} \tag{9.10}$$

The important content of these formulas is that the right-hand sides are *independent* of n: the equation that relates $\sigma_n(l-1)$ to $\sigma_n(l)$ is unchanged in all higher orders $n+p$ if l is replaced by $2^p l$. The notion of the set of σ_n interpreted as the nth-order step function approximant to the scaling *function* $\sigma(\tau)$ follows from this observation when τ is defined at level n as $l/2^{n+1}$:

$$\sigma_n(\tau) = \sigma_n(l) \qquad \text{for} \quad \tau \in (l/2^{n+1}, (l+1)/2^{n+1}) \tag{9.11}$$

The precise definition of T_n of (9.4) now follows. Throughout the rest of the discussion the subscript n on σ_n shall be implicitly understood. First, by (8.18)

$$x_2 = [1 - \sigma(2^n)]^{1/2} \tag{9.12}$$

Combining (8.19) and (8.20),

$$\sigma(2^n - 1) = \frac{1 - x_2}{1 + \sigma(2^{n+1} - 1)} \tag{9.13}$$

Now interpret (9.13) to determine the *new* (transformed under T) $\sigma(2^n - 1)$ by replacing x_2 in it by the square root of x_2^2 determined by the old $\sigma(2^n)$ as given in (9.12). Denoting transformed values by primes, we thus have

$$\text{(i)} \qquad \sigma'(2^n - 1) = \frac{1 - [1 - \sigma(2^n)]^{1/2}}{1 + \sigma(2^{n+1} - 1)} \tag{9.14}$$

This first equation required special treatment because of (8.17), now expressed by $r < 1$ in (9.9). As a (numerically, e.g.) systematic procedure, we save $(1 - x_2^2)$ $[= \sigma(2^n)]$ and $(1 - x_2)$ obtained by

$$(1 - x_2) = 1 - [1 - (1 - x_2^2)]^{1/2} = \frac{(1 - x_2^2)}{1 + [1 - (1 - x_2^2)]^{1/2}} \tag{9.15}$$

With these quantities known, we start with $r=1$ in (9.9), allowing $i=0, 1$, and rewrite it as

$$\text{(ii)} \qquad (1-x^2_{2i+2^{r+1}}) = (1-x^2_{2i}) + \sigma[2^{n-r}(2i+1)] \times [(1-x^2_{2i+2^r \bmod 2^{r+1}}) - (1-x^2_{2i})] \tag{9.16}$$

The $(1-x^2)$'s on the right-hand size are either $(1-x_0^2)=0$ or $(1-x_2^2)$ already saved. Thus, (9.16) determines the next two x_{2k}, $(1-x_4^2)$ and $(1-x_6^2)$, and the square root formula of (9.15) for general x_{2k} produces $(1-x_4)$ and $(1-x_6)$. Both of these quantities and the previous $(1-x_0)=0$ and $(1-x_2)$ now substituted in the right-hand side of (9.10) produces

$$\text{(iii)} \qquad \sigma'[2^{n-r}(2i+1)-1] = \frac{(1-x_{2i+2^{r+1}})-(1-x_{2i})}{(1-x_{2i+2^r \bmod 2^{r+1}})-(1-x_{2i})} \tag{9.17}$$

We now systematically increase $r=2,..., n$, the last step at $r=n$ determining $\sigma'(2i)$, and, in particular, $\sigma'(0)$. The only σ' *not* produced by (9.10) is $\sigma'(2^{n+1}-1)$ (i.e., $-\alpha^{-1}$). But this, of course, is where we inject the required nature of the algebraic singularity of g (i.e., quadratic) at this critical point, (8.22):

$$\text{(iv)} \qquad \sigma'(2^{n+1}-1) = [\sigma'(0)]^{1/2} \tag{9.18}$$

This completes the precise implementation of T, and is an algorithm that simultaneously produces all the x_{2k} together with all the σ's. At the heart of this dynamics is an insistence upon taking square roots, an inherently stabilizing operation. Verified to be convergent in low order, the difference delay intuition suggests a rapid convergence with order n and persisting stability. Numerically this is fully borne out to the degree that choosing the initial σ's randomly within (0, 1) always leads to convergence to the unique fixed point σ^*. Let us finish this discussion with the $n=0$ and $n=1$ delay models.

Setting $r=1$ in (9.9), we have, using (9.12) for x_2,

$$i=0: \qquad \sigma(2^{n-1}) = \frac{1-x_4^2}{1-x_2^2} \to x_4 = [1-\sigma(2^{n-1})\,\sigma(2^n)]^{1/2} \tag{9.19}$$

and

$$i=1: \qquad \sigma(2\cdot 2^n) = \frac{x_2^2-x_6^2}{x_2^2-1} \to x_6 = \{1-\sigma(2^n)[1-\sigma(3\cdot 2^{n-1})]\}^{1/2} \tag{9.20}$$

Substituting in (9.10), we have the transformation formula

$$\sigma'(2^{n-1}-1) = \frac{1-x_4}{1-x_2} = \frac{1-[1-\sigma(2^n)\,\sigma(2^{n-1})]^{1/2}}{1-[1-\sigma(2^n)]^{1/2}} \tag{9.21}$$

and

$$\sigma'(3\cdot 2^{n-1}-1)=\frac{x_2-x_6}{x_2-1}=\frac{\{1-\sigma(2^n)[1-\sigma(3\cdot 2^{n-1})]\}^{1/2}-[1-\sigma(2^n)]^{1/2}}{1-[1-\sigma(2^n)]^{1/2}} \tag{9.22}$$

From these results, we can see how (9.2) works out. Rationalizing numerator and denominator in (9.21), we have

$$\sigma'(2^{n-1}-1)=\sigma(2^{n-1})\frac{1+[1-\sigma(2^n)]^{1/2}}{1+[1-\sigma(2^n)\,\sigma(2^{n-1})]^{1/2}} \tag{9.23}$$

and an analogous result for (9.22). Notice that the ratio of the *sums* of roots is of order 1. As we write down the formulas for σ's at dyadically larger indices [i.e., larger r in (9.9) and (9.10)], the analogous ratio of sums of roots will be exponentially closer to 1 as more and more products of σ's (smaller than 1) appear subtracted from 1 within the square roots. That is, (9.5) will become exponentially accurate for σ's of larger r. Thus, as the order n increases, the new equations, all for the largest r $(=n)$ become exponentially closer to (9.5), and the dynamics "saturates." The first approximation to $n\to\infty$ is obtained by using just (i) of (9.14) for $l=2^n$ and for all other $l=1,\ldots,2^{n+1}-1$,

$$\sigma'(l-1)=\sigma(l) \tag{9.24}$$

Thus, the $r=0$ model dynamics is

$$\sigma_{l-1,t+1}=\sigma_{l,t}, \qquad l=1,\ldots,N-1,\ N+1,\ldots,2N-1 \tag{9.25a}$$

$$\sigma_{N-1,t+1}=\frac{1-(1-\sigma_{N,t})^{1/2}}{1+\sigma_{2N-1,t}} \tag{9.25b}$$

$$\sigma_{2N-1,t}=(\sigma_{0,t})^{1/2} \tag{9.25c}$$

where the time index t means $[\sigma_t]=T^t[\sigma_0]$. Let us compute the solution to (9.25). Write

$$X_t=\sigma_{2N-1,t}, \qquad Y_t=\sigma_{N-1,t} \tag{9.26}$$

By (9.25a)

$$\sigma_{N,t-1+N}=\sigma_{2N-1,t}=X_t \tag{9.27}$$

so that (9.25b) becomes

$$\sigma_{N-1,t+N}=Y_{t+N}=\frac{1-(1-X_t)^{1/2}}{1+X_{t+N-1}} \tag{9.28}$$

By (9.25a) again,

$$\sigma_{0,t+2N-1} = \sigma_{N-1,t+N} = Y_{t+N} \tag{9.29}$$

which by (9.25c), produces

$$X_{t+2N-1} = (Y_{t+N})^{1/2}$$

or

$$X_{t+N-1} = (Y_t)^{1/2} \tag{9.30}$$

Substituting (9.30) in (9.28) produces the system

$$X_{t+N-1} = (Y_t)^{1/2} \tag{9.31a}$$

$$Y_{t+N} = \frac{1-(1-X_t)^{1/2}}{1+(Y_t)^{1/2}} \tag{9.31b}$$

Next, defining

$$x_k = X_{kN} \tag{9.32a}$$

$$y_k = Y_{kN} \tag{9.32b}$$

and ignoring the -1 in X_{t+N-1} (order $1/N$; we are interested in $N \to \infty$), we have

$$x_{k+1} = (y_k)^{1/2} \tag{9.33}$$

$$y_{k+1} = \frac{1-(1-x_k)^{1/2}}{1+(y_k)^{1/2}}$$

The fixed point of (9.33) is just $x = \sigma_1$, $y = \sigma_0$ of (6.26) and is the $\boldsymbol{n=0}$ fixed point. The eigenvalues at the fixed point are, however, different from the $n=0$ **dynamics** of (i)–(iv). The system (9.33) is the $\boldsymbol{r=0}$ $n \to \infty$ dynamics. The eigenvalues of (9.33) at its fixed point are

$$\lambda_1 = -0.832289..., \qquad \lambda_2 = 0.688872... \tag{9.34}$$

Since $x_k - x^* \sim \lambda_1^k$, by (9.32), $X_{kN} - x^* \sim \lambda_1^k$, and so

$$X_{2Nk} - x^* \sim (\lambda_1^2)^k \tag{9.35}$$

That is, each $2N = 2^{n+1}$ iterates of (9.25) constituting one entire passage through all σ's, converges with eigenvalue λ_1^2, so that Λ of (9.6) is

$$\Lambda_0 = \lambda_1^2 = 0.692705... \tag{9.36}$$

Λ_0 turns out already to be an order 1% result for the full $n \to \infty$ dynamics.

The $r=1$ $n \to \infty$ dynamics follows by adding (9.21) and (9.22) for the quarter-way σ's to (9.14) for the half-way σ and (9.25a) for all others. Denoting $N=2^{n-1}$ here, and $X_t=\sigma_{4N-1,t}$ with analogous quantities for the other σ's, and the N-step values by lowercase letters as in (9.32), we obtain

$$\begin{aligned} x' &= u^{1/2} \\ y' &= \frac{[1-y(1-x)]^{1/2}-(1-y)^{1/2}}{1-(1-y)^{1/2}} \\ z' &= \frac{1-(1-y)^{1/2}}{1+u^{1/2}} \\ u' &= \frac{1-(1-yz)^{1/2}}{1-(1-y)^{1/2}} \end{aligned} \tag{9.37}$$

The fixed point is $x=\sigma_3$, $y=\sigma_2$, $z=\sigma_1$, $u=\sigma_0$ of the $n=1$ dynamics, which is given below (8.31). The spectrum of (9.37) at its fixed point is

$$\begin{aligned} \lambda_c &= 0.90623457\,e^{\pm i(\pi/2-0.051493628)} \\ \lambda_1 &= 0.83050196 \\ \lambda_2 &= -0.81607177 \end{aligned} \tag{9.38}$$

Since $N=2^{n-1}$, λ^4 are now the full passage through the eigenvalues of σ, and

$$\Lambda_1 = \lambda_c^4 = 0.6744698\,e^{\pm 0.2059745i} \tag{9.39}$$

The phase of Λ_1 means that convergence is a damped sinusoid of period ~ 30.50 (of full 2^{n+1} steps over the whole system). The modulus has changed $\sim 3\%$ from Λ_0. Indeed, the small departure of the phase of λ_c from $\pi/2$ is the very near agreement of λ_c^2 to the negative real λ_1 of (9.34). Also, $\lambda_1^2=0.68973\ldots$ in (9.38), to be compared to the subdominant $\lambda_2=0.688872$ of (9.34). That is, to surprising accuracy, the $r=0$ model faithfully follows the $r=1$ one. The actual $n \to \infty$ numerics are barely distinguishable (for n up to 8, or 512 σ's) from those of (9.37).

10. AFTERWORD AND CONCLUSIONS

In the last two sections we learned how the specification of the critical point singularity of the underlying dynamical map g determines the scaling function globally along the orbit. Since σ is invariant under smooth transformations, it is in particular almost a constant of the motion, since g is smooth away from its critical point, and hence σ everywhere is determined by the critical point singularity of g. The σ itself is a rich invariant encoding full knowledge of the temporal ordering along a strange set, and hence determining the refinement of a coarse-grained specification of the set by prologing the data x_t to successively larger ranges of t. Technically, as seen in the discussion surrounding (7.12), half of the information in σ is encoded in orbital eigenvalues; the other half of the information addresses the $O(1)$ coefficients of exponential quantities by determining the "finite"-size widths of the intervals obtained by the partitioning by the inverses under the presentation function. Let me be more precise.

By (3.3), the size of the intervals $\Delta^{(n)}(\varepsilon_n \cdots \varepsilon_1)$ is *estimated* by

$$\Delta^{(n)}(\varepsilon_n \cdots \varepsilon_1) \sim D(F_{\varepsilon_1} \cdots F_{\varepsilon_n}) \tag{10.1}$$

By choosing not x_0, but rather within $\Delta^{(n)}(0 \cdots 0)$ the periodic point

$$x_0^* = F_{\varepsilon_1} \cdots F_{\varepsilon_n}(x_0^*) \tag{10.2}$$

the derivative in (10.1) is effectively the eigenvalue of this orbit. This derivative in turn behaves as

$$\Delta^{(n)}(\varepsilon_n \cdots \varepsilon_1) \sim \sigma(\varepsilon_n \cdots \varepsilon_1)\, \sigma(\varepsilon_{n-1} \cdots \varepsilon_1) \cdots \sigma(\varepsilon_r \cdots \varepsilon_1) \cdots = k\sigma^n_{\text{eff}}(\varepsilon_n \cdots \varepsilon_1) \tag{10.3}$$

or,

$$\ln \sigma_{\text{eff}}(\varepsilon_n \cdots \varepsilon_1) \sim \frac{1}{n} \ln |D(F_{\varepsilon_1} \cdots F_{\varepsilon_n})| \tag{10.4}$$

It is precisely these asymptotic (in n) growth rate exponents that are determined by the slopes of $F_\varepsilon(x)$. Moreover, it is only this half of the information of σ that is "tested" by the thermodynamics. It follows that $f(\alpha)$ fails to encode the information of how the local linear segments of g are to be fitted together, and so is *far* from the full information invariantly available descriptive of the strange set (in addition to the complete loss of t-ordering information).

To put this differently, for $k = O(1)$ in (10.3), that equation asserts that the actual Δ's are of bounded variation from the Δ's *estimated* by $k = 1$. When we now reconsider the quotients leading to (7.12), we see that in the nth approximation to σ, asymptotically in m, the leading $m - n$ derivatives that determine σ_{eff} exactly cancel in numerator and denominator, so that σ is actually the quotient of the k's of (10.3) themselves. Thus, σ presents much finer information, inclusive of $O(1/n)$ terms, than the orbital eigenvalues and $f(\alpha)$, which determine information up to bounded variation only. This extra information allows orbit prolongation and attractor refinement not possible with the limited exponential information of (10.4).

Accordingly, a full and prescriptively useful end product of the "solution" of a dynamics $x_{t+1} = g(x_t)$ is the determination of its scaling function σ. But then, how can it be that σ is always determined from the equations that follow by eliminating the x_l from (8.15) and (8.16)? The answer to this query is the content of Section 6, specifically (6.10), which asserted that F_ε determines a period doubling fixed point. Inspection of the argument of Section 6 reveals that the "full" tree topology of inverses of F's as in Fig. 1, in contrast to an F of Fig. 4, is responsible for being led into period doubling. Indeed, an analysis like that of Section 6 applied to Fig. 4 would lead to a fixed-point dynamics of the golden mean renormalization group[11,12] and not that of period doubling. The analysis of Section 8 would then lead to σ's determined by the ratios of golden mean differences. I have presented the ideas of the golden mean scaling function in the context of circle maps elsewhere.[13] For the case of Fig. 4, the resulting σ is *not* that of a circle map, although of largely identical organization.

We thus see that the *topology* of intervals encoded in the topology of F (as in Fig. 4) determines the equations that fix σ. The golden mean topology *on a circle* is so much more important than that of Fig. 4 on an interval that I have not bothered here to present the scaling function theory for the problem whose thermodynamics is given by (5.4).

I have not done so because I have already worked out the theory for *circle map* topologies. That is, one can construct presentation functions on the circle. It turns out that there is a unique choice of the form of the golden mean fixed point that emerges whose scalings are identical to those of the original (i.e., not the fixed point) map for which E is expanding, and this choice is not that of Rand *et al.*[12] After F is known, the exact thermodynamic eigenvalue equation can be written down, which turns out to be subtly different from (5.4). There are a sufficient number of new ingredients in this circle map theory to make it inappropriate for this article. A sequel devoted to these new ideas is in preparation.

This brings us to a discussion of Sections 8 and 9, especially the latter. Section 8 is a culmination of a long effort to determine equations for the scaling function intrinsic to σ itself. (That is, not just determining σ as a detailed calculation available from the fixed-point dynamics g.) σ is the desired outcome of a calculation; g is the specification of the dynamics: it is the nature of the *orbits* of g that we want to know. Writing down g is writing down the *equations of motion*: since g is universal, at least we have written down a very generally applicable equation. But the real problem is to determine the *solution* to these equations, which are chaotic and derived from a highly nonlinear dynamics. As I have stressed, σ *is* the solution we seek in that it provides simple "genetic" building *knowledge* of the solution rather than a mindless enumeration of the ordered set $\{x_t\}$ of chaotically varying quantities. To generalize away from period doubling, we want to know how to generally write down equations for σ of a chaotic motion, and not just simulate the dynamics and "show" the solution in some complicated plot that merely reaffirms to the observer that the motion is chaotic—although at least not random. This is why I believe the ideas of the scaling function theory of Section 8 to be so important. But just so, I regard the dynamics in the space of scaling functions, Section 9, to be more important still.

As Section 9 stands, the σ dynamics is an "arbitrary" invention; as it might seem, merely a technical device to obtain the solution to the σ equations of Section 8. While true enough, I also believe otherwise.

As a mathematical point, the numerically observed good behavior of that dynamics (i.e., possessing a globally attracting fixed point), if proven, constitutes a proof of the uniqueness of the period doubling fixed point. (One would have to show that the solution for g just on the Cantor set is the restriction of an analytic function to prove existence by this line of thought. For the golden mean circle map fixed point, the "Cantor set" is now the entire circle, so a full solution is obtained now requiring proof of its analyticity.)

At least from this paper we know that nice scaling dynamics exists for a variety of highly nontrivial σ's. However, there is a conceptually deeper point: the equations of physics through the principles of inertia and causality are *local* (differential) equations of motion. Like the local (in t) dynamics $x_{t+1} = g(x_t)$, they fail to present in any transparent manner the inherently nonlocal "genetic" principle of a scaling function: the successive products of σ's as in (10.3) relate *distant* pieces of a solution to one another through common ancestors. What we must do is to implement that "change of variables" in the originally offered equations of motion to produce *dynamical equations* for σ itself. If we can do so, for σ's of just a few distinct values in some approximation, by easy calculation we can then compute the salient features of the solution and not the enumeration of the individually uninteresting x_t. Section 9 represents the first coming to grips with what such "intelligently" formulated dynamics should look like.

I want to conclude this paper with a conceptual analogy to (I hope) better illuminate the content of the last paragraph, and then exhibit an actual example of a true scalinglike dynamics drawn from the study of cellular automata.

Consider a cloudlike initial configuration of some fluid equation (a classical field theory). Imagine that the density of this configuration possesses rich scaling properties (e.g., a fixed spatial scale exponent over many decades.) Moreover, imagine that at successive moments of time it also possesses these scaling properties, although possibly variable in time. From this we should surmise that the instantaneous velocities should also possess similar scaling properties. Imagine that these scalings are easily specified, that is, we have discerned in this complex spatial object some prescriptive rules that if iterated would construct it. Now let us contemplate how we advance this structure in time. By the locality of the field equations we must *actually* spin out this iterative construction in order to provide the equations with the sort of initial data they require. Now we can advance the structure a step ahead in time. But what do we now have? Simply an immense list of local density and velocity values of high local irregularity. Of course, if we possess a good algorithm, we could now from this new pabulum of data again discern the scaling information—perhaps evolved—that we knew about anyway. This is obviously a foolish double regress. Since our informed understanding lay in the scaling description, we should obviously have transcribed our "true" local dynamics into one pertinent to these scalings, rather than mount a numerical program that strains the most powerful machines we possess. That is, the solution in the usual sense of our local field theories is apt to be a mindless enterprise when the solutions happen not to be simple. In this sense, our theories, while "true," are useful only to God, which seems not to be the hallmark of what humans adjudge to be truth.

As our last heuristic example, consider the time-dependent block probabilities of a one-dimension, nearest-neighbor, two-state-per-site cellular automaton. Denoting these probabilities in an nth approximation by $P_n(\varepsilon_m \cdots \varepsilon_1 \varepsilon_0)$, we will cast an analogy between the P_n and Δ_n. As with the Δ_n, the $P_n(\varepsilon_m \cdots \varepsilon_1 \varepsilon_0)$ are assembled from Markov transition probabilities $\sigma_n(\varepsilon_n \cdots \varepsilon_0)$ linking the 2^n nodes labeled by $\varepsilon_n \cdots \varepsilon_1$ of a strictly probabilistic Markov graph (the sum of the transition probabilities out of a node sums to one). Assigning nodal probabilities $\Pi_n(\varepsilon_n \cdots \varepsilon_1)$ determined by the stationarity of the process then produces the nth-order probabilities

$$P_n(\varepsilon_n \cdots \varepsilon_0) = \sigma_n(\sigma_n \cdots \varepsilon_0)\, \Pi_n(\varepsilon_n \cdots \varepsilon_1) \tag{10.5}$$

The point of this construction is that with the σ_n randomly assigned, these P_n are then a random *a priori* set of probabilities satisfying the Kolmogorov consistency conditions (i.e., that these P_n summed over right or left ε's consistently produces the lower-order P_{n-r} probabilities). The minimal-information *extension* of these P_n is now simply

$$\begin{aligned} &P_{n+r}(\varepsilon_{n+r},..., \varepsilon_{n+1}, \varepsilon_n,..., \varepsilon_0) \\ &\quad = \sigma_n(\varepsilon_{n+r},..., \varepsilon_r) \cdots \sigma_n(\varepsilon_{n+1},..., \varepsilon_1)\, P_n(\varepsilon_n,..., \varepsilon_0) \end{aligned} \tag{10.6}$$

so that the extensions P_{n+r} are constructed in precisely the same way as the $\Delta^{(n)}$ of (10.3). The important idea of Gutowitz *et al.*[14] (in addition to the Baysean extension method of the P_{n+r} equivalent to the Markov diagrammatic method just presented) is that the *dynamical* action of the automaton relates the P_n at time $t+1$ as an appropriate (depending upon the rule of the automaton) function of the $P_{n+2}(\varepsilon_{n+1}, \varepsilon_n, \ldots, \varepsilon_0, \varepsilon_{-1})$ for nearest neighbor rules (whence the additional ε_{n+1} and ε_{-1}). By (10.6) this becomes a dynamical rule for the evolution of the basic entities entities σ_n, Thus, in formal analogy, the kind of dynamics discussed in Section 9 is the actual temporal dynamics of these automata. That is, the idea of a dynamics *not* for the "obvious" variables $P(\cdots \varepsilon_m \cdots \varepsilon_0 \cdots)$, but rather for the "scaling" elaborative variables σ_n is the actual dynamics of these systems, so that the scheme of evolving "evolutionary" variables, with no requirement of approach to, say, a fixed point, is here realized.

Thus, in conclusion, I hope to have exposed some glimmers of a new program of dynamics for problems in which our accustomed partial differential fied equations lead us into a hopeless morass of boring numerical simulation. My examples to date are indeed too special, but perhaps suggestively illuminating.

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